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# Nori's fundamental group-scheme

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And when we were children, staying at the archduke's,
my cousin's, he took me out on a sled,
and I was frightened. He said, Marie,
Marie, hold on tight. And down we went.
T.S. ELIOT, *The waste land*, 13-16

# Contents

# Introduction

As is widely known, Galois theory has strong analogies with the topological theory of the fundamental group. If $X$ is a nice enough topological space, finite Galois coverings of $X$ form a projective system, and the limit of the automorphism groups of these coverings is the profinite completion of the fundamental group. If $k$ is a field and $k^{\mathrm{s}}$ a separable closure, the set of finite Galois subextensions of $k_{\mathrm{s}}/k$ forms a projective system and $\mathrm{Gal}(k_{\mathrm{s}}/k)$ is the limit of the automorphism groups of these finite subextensions.

Essentially, we can recover Galois groups and fundamental groups from their categories of finite quotients, which have a nice description in terms of automorphism groups of Galois extensions and Galois coverings. Grothendieck in [SGA1] had the idea of describing a profinite group $G$ using the category $\mathcal{C}$ of finite sets with a continuous action of $G$ (the *Galois category* of $G$) rather than the one of its finite quotients. In fact, $G$ is the group of automorphism of the forgetful functor $\omega : \mathcal{C} \to \mathrm{Set}$, called *fibre functor*.

From this point of view, the Galois category of $\mathrm{Gal}(k_{\mathrm{s}}/k)$ has a nice interpretation: it is the opposite of the category of étale $k$-algebras (as proved in Theorem 2.20). On the other hand, the Galois category of the profinite completion of the fundamental group is simply the category of finite coverings. In algebraic geometry, these two points of view merge in the concept of finite étale covering, and Grothendieck defined the *étale fundamental group* as the profinite group associated to the Galois category of finite étale coverings of a scheme.

Meanwhile Saavedra Rivano in [Saa72], using an idea of Tannaka and Krein, developed under the direction of Grothendieck the theory of Tannakian categories (with an error lately corrected by Deligne): he proved that an affine group-scheme $G$ can be recovered from the category of its representations $\mathrm{Rep}_k G$ with the forgetful functor $\omega : \mathrm{Rep}_k G \to \mathrm{Vect}_k$ (also called fibre functor), and described exactly which categories arise in this way, the neutral Tannakian categories.

Ten years later, Nori in [Nor82] merged these ideas to define a new fundamental group, using principal bundles with finite fiber instead of finite étale coverings and Tannakian categories instead of Galois categories. This different approach led to an invariant with a richer structure of affine group-scheme, with the crucial advantage of taking into account those cases in which geometry does not reflect enough the underling richer algebraic structure: for example, when $k$ has positive characteristic. For a reduced and connected scheme $X$ with a rational point $x_0$, he defined an affine group-scheme $\pi_1^N(X, x_0)$ with the property that morphisms $\pi_1^N(X, x_0) \to G$, where $G$ is a finite group-scheme, correspond to $G$-torsors over the base scheme. Moreover he proved that, when $X$ is complete, the Tannakian category of $\pi_1^N(X, x_0)$ corresponds to the category of vector bundles over $X$ with a particular condition of finiteness, the essentially finite vector bundles.

We finally come to the present days: in [BV12], Borne and Vistoli made a broad generalization of Nori's work. They replaced the base scheme with a fibered category and group-schemes with gerbes, they removed the assumption that $X$ has a rational point and they relaxed the completeness hypothesis asking $X$ only to be *pseudo-proper*: $X$ has to be quasi-compact and, for every locally free sheaf of finite rank $E$ over $X$, to satisfy $\mathrm{H}^0(X, E) < +\infty$. Moreover, they took a more direct approach to the proof of the Tannakian interpretation of the fundamental group.

In the present thesis, we follow the work of Borne and Vistoli to define the fundamental group-scheme of a geometrically connected and geometrically reduced base scheme $X$ with a fixed rational point, and we show the Tannakian interpretation when $X$ is pseudo-proper. The first five chapters are mainly devoted to the construction of the tools we will use in the last one, the most interesting: it contains Borne and Vistoli's proof of the Tannakian interpretation in our setting of schemes and group-schemes.

We made an effort to keep the thesis as self contained as possible. We assume the reader knows the basics of scheme theory (the first three chapters of [Liu02]) and of commutative algebra (the entire [AM69]), plus some notions of category theory.

# Chapter 1

# Category theory

We begin by briefly recalling some notions of category theory. We assume the reader is familiar with the notions of category, functor and natural transformation. Otherwise, [Bor94] is a good reference.

## 1.1 Equivalences of categories

**Definition 1.1.** Let $\mathcal{F}, \mathcal{F}' : \mathcal{A} \to \mathcal{B}$ be functors. A morphism of functors (i.e. a natural transformation) $\alpha : \mathcal{F} \to \mathcal{F}'$ is an *isomorphism of functors* if, for every object $X \in \mathcal{A}$, $\alpha_X : \mathcal{F}X \to \mathcal{F}'X$ is an isomorphism.

**Definition 1.2.** A functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is an *equivalence of categories* if there exists a functor $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ and isomorphisms of functors $\mathcal{G} \circ \mathcal{F} \simeq \mathrm{id}_{\mathcal{A}}$, $\mathcal{F} \circ \mathcal{G} \simeq \mathrm{id}_{\mathcal{B}}$. In this case, $\mathcal{G}$ is called a *quasi-inverse* of $\mathcal{F}$.

**Example 1.3.** Let $(X, x)$ be a pointed, connected topological space with a universal covering $(U, u) \to (X, x)$, we have a natural identification $\pi_1(X, x) = \mathrm{Aut}(U/X)$. Call $\mathrm{Cov}_{(X,x)}$ the category of connected coverings $(E, e) \to (X, x)$, and $\mathrm{S}_{\pi_1(X,x)}$ the category of subgroups of $\pi_1(X, x)$.

If $(E, e) \to (X, x)$ is a connected covering, $\pi_1(E, e)$ is naturally a subgroup of $\pi_1(X, x)$. This defines a functor $\pi_1 : \mathrm{Cov}_{(X,x)} \to \mathrm{S}_{\pi_1(X,x)}$. On the other hand, if $G \subseteq \pi_1(X, x)$ is a subgroup, $(U/G, [u])$ is a connected covering of $(X, x)$, and this association extends to a functor $\mathcal{Q} : \mathrm{S}_{\pi_1(X,x)} \to \mathrm{Cov}_{(X,x)}$. The composition $\pi_1 \circ \mathcal{Q}$ is the identity of $\mathrm{S}_{\pi_1(X,x)}$, but $\mathcal{Q} \circ \pi_1$ is not the identity of $\mathrm{Cov}_{(X,x)}$: if $(E, e)$ is a covering, it is isomorphic to $(U/\pi_1(E, e), [u])$, but in general they are not equal. However, this isomorphism ensures that $\pi_1$ is an equivalence of categories between $\mathrm{Cov}_{(X,x)}$ and $\mathrm{S}_{\pi_1(X,x)}$.

**Definition 1.4.** Let $\mathcal{F}, \mathcal{F}' : \mathcal{A} \to \mathcal{B}$, $\mathcal{G}, \mathcal{G}' : \mathcal{B} \to \mathcal{C}$ be functors, and $\alpha : \mathcal{F} \to \mathcal{F}'$, $\beta : \mathcal{G} \to \mathcal{G}'$ morphisms of functors.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\mathcal{F}"] \arrow[r, bend right, "\mathcal{F}'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[r, bend left, "\mathcal{G}"] \arrow[r, bend right, "\mathcal{G}'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{C} \end{tikzcd}$$

For every object $X$ of $\mathcal{A}$, the diagram

$$\begin{array}{ccc} \mathcal{G}\mathcal{F}X & \xrightarrow{\beta_{\mathcal{F}X}} & \mathcal{G}'\mathcal{F}X \\ \Big\downarrow{\scriptstyle \mathcal{G}\alpha_X} & & \Big\downarrow{\scriptstyle \mathcal{G}'\alpha_X} \\ \mathcal{G}\mathcal{F}'X & \xrightarrow{\beta_{\mathcal{F}'X}} & \mathcal{G}'\mathcal{F}'X \end{array}$$

is commutative thanks to the naturality of $\beta$. The composition defines a morphism of functors $\mathcal{G}\mathcal{F} \to \mathcal{G}'\mathcal{F}'$ called the *Godement product* of $\alpha$ and $\beta$, it is indicated by $\beta * \alpha$. Naturality of $\beta * \alpha$ comes from the fact that, for every morphism $f : A \to A'$, the diagram

$$\begin{array}{ccccc} \mathcal{G}\mathcal{F}A & \xrightarrow{\mathcal{G}\alpha_A} & \mathcal{G}\mathcal{F}'A & \xrightarrow{\beta_{\mathcal{F}'A}} & \mathcal{G}'\mathcal{F}'A \\ \Big\downarrow{\scriptstyle \mathcal{G}\mathcal{F}f} & & \Big\downarrow{\scriptstyle \mathcal{G}\mathcal{F}'f} & & \Big\downarrow{\scriptstyle \mathcal{G}'\mathcal{F}'f} \\ \mathcal{G}\mathcal{F}A' & \xrightarrow{\mathcal{G}\alpha_{A'}} & \mathcal{G}\mathcal{F}'A' & \xrightarrow{\beta_{\mathcal{F}'A'}} & \mathcal{G}'\mathcal{F}'A' \end{array}$$

commutes thanks to naturality of $\alpha$ and $\beta$.

For the sake of brevity, we shall often write $\beta * \mathcal{F}$ for $\beta * \mathrm{id}_{\mathcal{F}}$ and $\mathcal{G} * \alpha$ for $\mathrm{id}_{\mathcal{G}} * \alpha$.

**Definition 1.5.** Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ be functors. We will say that $\mathcal{G}$ is a *left adjoint* to $\mathcal{F}$ (or that $\mathcal{F}$ is a *right adjoint* to $\mathcal{G}$) if, for every $A$ in $\mathcal{A}$ and $B$ in $\mathcal{B}$, there exists bijections

$$\theta_{A,B} : \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}B, A) \simeq \mathrm{Hom}_{\mathcal{B}}(B, \mathcal{F}A)$$

functorial both in $A$ and in $B$.

**Example 1.6.** The inclusion $\mathrm{Ab} \hookrightarrow \mathrm{Grp}$ of the category of abelian group in the category of abelian groups has a left adjoint: it is the functor $\mathrm{Grp} \to \mathrm{Ab}$ sending a group $G$ to its abelianization $G_{\mathrm{ab}}$.

**Proposition 1.7.** *Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ be functors. The following are equivalent.*

- *$\mathcal{G}$ is a left adjoint to $\mathcal{F}$.*
- *There exist natural transformations $\eta : \mathrm{id}_{\mathcal{B}} \to \mathcal{F} \circ \mathcal{G}$ and $\varepsilon : \mathcal{G} \circ \mathcal{F} \to \mathrm{id}_{\mathcal{A}}$ such that*

$$(\mathcal{F} * \varepsilon) \circ (\eta * \mathcal{F}) = \mathrm{id}_{\mathcal{F}}, \ (\varepsilon * \mathcal{G}) \circ (\mathcal{G} * \eta) = \mathrm{id}_{\mathcal{G}} .$$

*Proof.* We are only interested in how the first condition implies the second. For a complete proof, see [Bor94, Theorem 3.1.5].

Hence, suppose that $\mathcal{G}$ is a left adjoint to $\mathcal{F}$. There exist bijections

$$\theta_{A,B} : \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}B, A) \simeq \mathrm{Hom}_{\mathcal{B}}(B, \mathcal{F}A)$$

functorial in $A, B$ for every $A$ and $B$. For every $A, B$, call $\eta_B$ the morphism

$$\theta_{\mathcal{G}B,B}(\mathrm{id}_{\mathcal{G}B}) \in \mathrm{Hom}(B, \mathcal{F}\mathcal{G}B)$$

and $\varepsilon_A$ the morphism

$$\theta^{-1}_{A,\mathcal{F}A}(\mathrm{id}_{\mathcal{F}A}) \in \mathrm{Hom}(\mathcal{G}\mathcal{F}A, A).$$

If $f : B \to B'$ is a morphism, the diagram

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{G}B} \arrow[d, mapsto] & \mathrm{Hom}(\mathcal{G}B, \mathcal{G}B) \arrow[r] \arrow[d] & \mathrm{Hom}(B, \mathcal{F}\mathcal{G}B) \arrow[d] & \eta_B \arrow[d, mapsto] \\
\mathcal{G}f & \mathrm{Hom}(\mathcal{G}B, \mathcal{G}B') \arrow[r] & \mathrm{Hom}(B, \mathcal{F}\mathcal{G}B') & \mathcal{F}\mathcal{G}f \circ \eta_B = \eta_{B'} \circ f \\
\mathrm{id}_{\mathcal{G}B'} \arrow[u, mapsto] & \mathrm{Hom}(\mathcal{G}B', \mathcal{G}B') \arrow[r] \arrow[u] & \mathrm{Hom}(B', \mathcal{F}\mathcal{G}B') \arrow[u] & \eta_{B'} \arrow[u, mapsto]
\end{tikzcd}$$

is commutative thanks to naturality of $\theta$ and hence shows naturality of $\eta$. Naturality of $\varepsilon$ is analogous. We want now to check $(\varepsilon * \mathcal{G}) \circ (\mathcal{G} * \eta) = \mathrm{id}_{\mathcal{G}}$:

for every object $B$ in $\mathcal{B}$ we need to show $\varepsilon_{\mathcal{G}B} \circ \mathcal{G}\eta_B = \mathrm{id}_{\mathcal{G}B}$.

$$\begin{array}{ccccc} \mathrm{Hom}(\mathcal{G}B, \mathcal{G}B) & \xrightarrow{\theta} & \mathrm{Hom}(B, \mathcal{F}\mathcal{G}B) & \xrightarrow{\mathcal{G}} & \mathrm{Hom}(\mathcal{G}B, \mathcal{G}\mathcal{F}\mathcal{G}B) \\ \mathrm{id}_B & \longmapsto & \eta_B & \longmapsto & \mathcal{G}\eta_B \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{F}\mathcal{G}B, \mathcal{F}\mathcal{G}B) & \xrightarrow{\theta^{-1}} & \mathrm{Hom}(\mathcal{G}\mathcal{F}\mathcal{G}B, \mathcal{G}B) \\ \mathrm{id}_{\mathcal{F}\mathcal{G}B} & \longmapsto & \varepsilon_{\mathcal{G}B} \end{array}$$

Now, naturality of $\theta^{-1}$ in the first argument implies that

$$\mathrm{id}_{\mathcal{G}B} = \theta^{-1}(\eta_B) = \theta^{-1}(\mathrm{id}_{\mathcal{F}\mathcal{G}B} \circ \eta_B) = \theta^{-1}(\mathrm{id}_{\mathcal{F}\mathcal{G}B}) \circ \mathcal{G}\eta_B = \varepsilon_{\mathcal{G}B} \circ \mathcal{G}\eta_B.$$

Analogously, to prove that $(\mathcal{F} * \varepsilon) \circ (\eta * \mathcal{F}) = \mathrm{id}_{\mathcal{F}}$ we take an object $A$ in $\mathcal{A}$, naturality of $\theta$ in the second argument implies that

$$\mathrm{id}_{\mathcal{F}A} = \theta(\varepsilon_A) = \theta(\varepsilon_A \circ \mathrm{id}_{\mathcal{G}\mathcal{F}A}) = \mathcal{F}\varepsilon_A \circ \theta(\mathrm{id}_{\mathcal{G}\mathcal{F}A}) = \mathcal{F}\varepsilon_A \circ \eta_{\mathcal{F}A}.$$

□

To prove that a functor is an equivalence of categories one should always construct explicitly a quasi-inverse, but this is often rather difficult and tedious. When one wants to prove that a function is a bijection, it is often simpler to show that it is injective and surjective than to construct explicitly the inverse: we would like to have some similar criterion for functors between categories. Moreover, even if we know that a functor is an equivalence and we need a quasi-inverse, we have in general a lot of choices: taking an arbitrary quasi-inverse may result in having naturality problems. For example, if $\mathcal{F}$ and $\mathcal{G}$ are quasi-inverses, the definitions give us two in general different natural transformations $\mathcal{F}\mathcal{G}\mathcal{F} \to \mathcal{F}$, which in general will not be equal. The following proposition resolves these problems.

**Proposition 1.8.** *Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a functor. The following are equivalent.*

- *$\mathcal{F}$ is an equivalence of categories.*

- *$\mathcal{F}$ has a left adjoint $\mathcal{G}$ such that the two induced natural transformations $\mathrm{id}_{\mathcal{B}} \to \mathcal{F} \circ \mathcal{G}$, $\mathcal{G} \circ \mathcal{F} \to \mathrm{id}_{\mathcal{A}}$ are isomorphisms.*
- *$\mathcal{F}$ has a right adjoint $\mathcal{G}$ such that the two induced natural transformations $\mathrm{id}_{\mathcal{A}} \to \mathcal{G} \circ \mathcal{F}$, $\mathcal{F} \circ \mathcal{G} \to \mathrm{id}_{\mathcal{B}}$ are isomorphisms.*
- *$\mathcal{F}$ is fully faithful and essentially surjective.*

*Proof.* [Bor94, Proposition 3.4.3]. □

## 1.2 Representable functors

When $\mathcal{C}, \mathcal{C}'$ are categories, we have a third category

$$\mathrm{Hom}(\mathcal{C}, \mathcal{C}')$$

whose objects are functors from $\mathcal{C}$ to $\mathcal{C}'$ and whose arrows are natural transformations between functors.

There is a natural way of embedding a category $\mathcal{C}$ in the category

$$\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

I.e., there is a natural way of thinking to an object of $\mathcal{C}$ as a contravariant functor from $\mathcal{C}$ to Set that preserves all the information of $\mathcal{C}$. Fix an object $X \in \mathrm{Obj}\,\mathcal{C}$: we can define a functor by sending an arbitrary object $T$ of $\mathcal{C}$ to the set $\mathrm{Hom}_{\mathcal{C}}(T, X)$ of arrows $T \mapsto X$, the "$T$-points" of $X$, and by sending a morphism $f : S \to T$ in $\mathcal{C}$ to the function $\mathrm{Hom}_{\mathcal{C}}(T, X) \to \mathrm{Hom}_{\mathcal{C}}(S, X)$ induced by composition with $f$. It is immediate to check that these maps preserve identity and composition, hence they define a functor $\mathrm{h}_X : \mathcal{C}^{\mathrm{op}} \to$ Set called the "functor of points" of $X$.

**Definition 1.9.** A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ isomorphic to $\mathrm{h}_X$ for some $X$ is called a representable functor, and we say that it is represented by $X$.

**Example 1.10.** If we consider the category $\mathcal{C}$ of pointed topological spaces with arrows given by continuous maps up to homotopy, the fundamental group $(X, x_0) \mapsto \pi_1(X, x_0)$ is a representable functor $\mathcal{C} = (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Set}$: it is represented by $(\mathrm{S}^1, s)$.

**Example 1.11.** Consider the category $\mathrm{Sch}\,/k$ of schemes over $k$, and take the functor $\Gamma$ of global sections $T \mapsto \mathrm{H}^0(T, \mathcal{O}_T)$. Since the scheme $\mathbb{A}^1_k = \mathrm{Spec}\,k[x]$ is affine, maps from a scheme $T$ into $\mathbb{A}^1_k$ are in bijective correspondence with homomorphisms of algebras $k[x] \to \mathrm{H}^0(T, \mathcal{O}_T)$. To choose such a map we only need to choose the image of $x$, and $k[x]$ is freely generated as a $k$ algebra by $x$, hence we only need to choose any global section of $T$. This means that $\mathbb{A}^1_k$ represents $\Gamma$. Similarly, one can show that the functor $\Gamma^n$ sending a scheme $T$ to $n$-uples of global sections is represented by $\mathbb{A}^n_k$.

**Example 1.12.** Consider an invertible sheaf $L$ on a scheme $T$ and $n+1$ sections $(s_0, \ldots, s_n) \in \mathrm{H}^0(T, L)^{n+1}$. We say that the vector $(L, s_0, \ldots, s_n)$ is never zero if, for every $p \in T$,

$$(s_0(p), \ldots, s_n(p)) \neq (0, \ldots, 0) \in L(p)^{n+1}.$$

Now, take the functor $\mathrm{proj}_n$ from $\mathrm{Sch}/k^{\mathrm{op}}$ to Set sending a scheme $T$ to the set of never zero vectors $(L, s_0, \ldots, s_n)$ up to equivalence, where we say that $(L, s_0, \ldots, s_n) \sim (L', s'_0, \ldots, s'_n)$ if there exists an isomorphism $\varphi : L \to L'$ such that $(\varphi(s_0), \ldots, \varphi(s_n)) = (s'_0, \ldots, s'_n)$. If $f : S \to T$ is a morphism and $(L, s_0, \ldots, s_n)$ is never zero on $T$, the vector $\mathrm{proj}_n(f)(L, s_0, \ldots, s_n) = (f^*L, f^*s_0, \ldots, f^*s_n)$ is never zero on $S$. We claim that $\mathbb{P}^n$ represents $\mathrm{proj}_n$.

Take $(s_0, \ldots, s_n) \in \mathrm{H}^0(T, L)^{n+1}$, if it is never zero then $T = \bigcup_i T_{s_i}$. We may cover $\mathbb{P}^n$ with open affine sets, $\mathbb{P}^n = \bigcup_i \operatorname{Spec} k\left[\frac{x_j}{x_i}\right]_{0 \le j \le n}$ and define

$$f_i : T_{s_i} \to \operatorname{Spec} k\left[\frac{x_j}{x_i}\right]_{0 \le j \le n}$$

by $x_j/x_i \mapsto s_j/s_i$. In fact, $s_j/s_i$ is a section of $\mathrm{H}^0(T_{s_i}, \mathcal{O}_T)$: we may define it locally using a trivialization of $L$, observe that it does not depend on the trivialization and use this fact to glue. Moreover, the morphisms $f_0, \ldots, f_n$ glue to a global morphism $f : T \to \mathbb{P}^n$. If $\varphi : L \to L'$ is an isomorphism of invertible sheaves, $\frac{s_j}{s_i} = \frac{\varphi(s_j)}{\varphi(s_i)}$ on $T_{s_i}$: if $\mathcal{O}_p \to L_p$ is a trivialization, the composition $\mathcal{O}_p \to L_p \xrightarrow{\varphi} L'_p$ is a trivialization, too, and hence $\frac{s_j}{s_i} = \frac{\varphi(s_j)}{\varphi(s_i)}$ on $p$ by definition. This implies that $f$ is well defined. On the other hand, take a morphism $f : T \to \mathbb{P}^n$, and consider the vector $(f^*\mathcal{O}(1), f^*x_0, \ldots, f^*x_n)$: it is never zero because for every $p \in T$ there exists $i$ such that $x_i(f(p)) \neq 0 \in \mathcal{O}(1)(f(p))$.

These two constructions are inverses: let the vector $(L, s_0, \ldots, s_n)$ be never zero and $f : T \to \mathbb{P}^n$ the associated morphism. Since both $s_i$ and $f^*x_i$ are never zero on $T_{s_i}$, the assignment $s_i \mapsto x_i$ defines an isomorphism of invertible sheaves $\varphi_i : L|_{T_{s_i}} \to f^*\mathcal{O}(1)|_{T_{s_i}}$. Thanks to the definition of $f$, these isomorphisms glue to a global isomorphism $\varphi : L \to f^*\mathcal{O}(1)$ such that $\varphi(s_i) = f^*x_i$. On the other hand, if $f : T \to \mathbb{P}^n$ is a morphism, the morphism defined by the vector $(f^*\mathcal{O}(1), f^*x_0, \ldots, f^*x_n)$ is exactly $f$, as can be seen locally.

## 1.3 The Yoneda Lemma

For every object of $X$ of $\mathcal{C}$, we have given an object $\mathrm{h}_X$ of

$$\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

We shall see this assignment completes to a functor

$$\mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}).$$

The Yoneda Lemma (in its weak form) says that this functor is fully faithful, allowing us to work with $\mathrm{h}_X$ instead of $X$.

Now, given a morphism $f : X \to Y$ in $\mathcal{C}$ and an object $T$ of $\mathcal{C}$, composition with $f$ defines a morphism $\mathrm{h}_f(T) : \mathrm{h}_X(T) = \mathrm{Hom}(T, X) \to \mathrm{h}_Y(T) = \mathrm{Hom}(T, Y)$, and if $g : S \to T$ is another morphism in $\mathcal{C}$, this yields to a commutative diagram

$$\begin{array}{ccc} \mathrm{h}_X(T) & \xrightarrow{\mathrm{h}_f(T)} & \mathrm{h}_Y(T) \\ \downarrow{\scriptstyle \mathrm{h}_X(g)} & & \downarrow{\scriptstyle \mathrm{h}_Y(g)} \\ \mathrm{h}_X(S) & \xrightarrow{\mathrm{h}_f(S)} & \mathrm{h}_Y(S) \end{array}$$

This means that a morphism $f : X \to Y$ induces a natural transformation $\mathrm{h}_f : \mathrm{h}_X \to \mathrm{h}_Y$, and one can easily verify that this makes $\mathrm{h}$ a functor from $\mathcal{C}$ to $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$.

**Theorem 1.13** (the Yoneda Lemma - weak form)**.** *The functor* $\mathrm{h}$ *is fully faithful.*

*Proof.* The fact that h is fully faithful simply means that arrows $X \to Y$ are in bijective correspondence with natural transformations $\mathrm{h}_X \to \mathrm{h}_Y$. We have already associated a natural transformation to a morphism in $\mathcal{C}$, now we do the converse.

Let $\mathfrak{T} : \mathrm{h}_X \to \mathrm{h}_Y$ be a natural transformation. Since $h_f(\mathrm{id}_X) = f \circ \mathrm{id}_X = f$ and we want to define an inverse to $f \mapsto \mathrm{h}_f$, it is a good idea to take $\mathfrak{T}(\mathrm{id}_X) : X \to Y$. We need to show that these two constructions are inverses.

Firstly, as we have already seen, if we take a morphism $f : X \to Y$ we have $h_f(\mathrm{id}_X) = f$.

On the other hand, given a natural transformation $\mathfrak{T}$, we need to check that $\mathrm{h}_{\mathfrak{T}(\mathrm{id}_X)} = \mathfrak{T}$. Hence, take $S \in \operatorname{Obj} \mathcal{C}$ and consider

$$\mathrm{h}_{\mathfrak{T}(\mathrm{id}_X)}(S) : \mathrm{h}_X(T) \to \mathrm{h}_Y(S)$$

Let $s \in \mathrm{h}_X(S)$ be a morphism $S \to X$: since $\mathfrak{T}$ is a natural transformation, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{h}_X(X) & \xrightarrow{\mathfrak{T}_X} & \mathrm{h}_Y(X) \\ \downarrow{\scriptstyle \mathrm{h}_X(s)} & & \downarrow{\scriptstyle \mathrm{h}_Y(s)} \\ \mathrm{h}_X(S) & \xrightarrow{\mathfrak{T}_S} & \mathrm{h}_Y(S) \end{array}$$

Applying it to $\mathrm{id}_X$, we get the equality

$$s \circ \mathfrak{T}_X(\mathrm{id}_X) = \mathfrak{T}_S(s \circ \mathrm{id}_X) = \mathfrak{T}_S(s)$$

and this simply means $\mathrm{h}_{\mathfrak{T}(\mathrm{id}_X)} = \mathfrak{T}$. $\square$

**Example 1.14.** Take the open subscheme $\mathbb{A}_k^{n+1} \setminus \{0\} \subseteq \mathbb{A}_k^{n+1}$. Using Example 1.11, one can see that it represents the functor

$$T \mapsto \{(s_0, \ldots, s_n) \mid s_i \in \mathrm{H}^0(T, \mathcal{O}_T), \forall p \in T : (s_0(p), \ldots, s_n(p)) \neq (0, \ldots, 0)\}$$

There is an obvious natural transformation $\mathrm{h}_{\mathbb{A}_k^{n+1} \setminus \{0\}} \to \mathrm{proj}_n$ sending $(s_0, \ldots, s_n)$ to $(\mathcal{O}_T, s_0, \ldots, s_n)$. The Yoneda Lemma tells us that this corresponds to a morphism $\mathbb{A}_k^{n+1} \setminus \{0\} \to \mathbb{P}_k^n$.

From now on, with abuse of notation, we will confuse $X$ with $\mathrm{h}_X$, using $X(S)$ for $\mathrm{Hom}_{\mathcal{C}}(S, X)$.

There is a more general version of the Yoneda Lemma, asking only to the first functor to be representable. The proof is analogous to the one of the weak form of the lemma.

**Theorem 1.15** (the Yoneda Lemma)**.** *Given a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and an object $X$ of $\mathcal{C}$, there is a bijective correspondence between natural transformations $\mathfrak{T} : \mathrm{h}_X \to F$ and $F(X)$ induced by $\mathfrak{T} \mapsto \mathfrak{T}(\mathrm{id}_X)$.*

*Proof.* Firstly note that, if $F = \mathrm{h}_Y$, then $F(X) = \mathrm{Hom}(X, Y)$, and we get the weak form of the Yoneda Lemma.

If $\xi \in F(X)$ and $S$ is an object of $\mathcal{C}$, there is a map $h_X(S) = \mathrm{Hom}_{\mathcal{C}}(S, X) \to F(S)$ sending $s : S \to X$ to $s^*\xi$. This defines a natural transformation $\mathrm{h}_\xi$: if I take a morphism $f : T \to S$ then $(s \circ f)^*\xi = f^*s^*\xi$, hence the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{h}_X(S) & \xrightarrow{\cdot^*\xi} & F(S) \\ \big\downarrow{\scriptstyle \mathrm{h}_X(f)} & & \big\downarrow{\scriptstyle F(f)} \\ \mathrm{h}_X(T) & \xrightarrow{\cdot^*\xi} & F(T) \end{array}$$

Conversely, given a natural transformation $\mathfrak{T} : \mathrm{h}_X \to F$, $\mathfrak{T}_X(\mathrm{id}_X)$ is an element of $F(X)$.

Firstly, we need to check that, given $\xi \in F(X)$, then $\mathrm{h}_\xi(\mathrm{id}_X) = \xi$, but this is obvious because $\mathrm{id}_X^*\, \xi = \xi$.

Secondly, we need to check that, given a natural transformation $\mathfrak{T}$ : $\mathrm{h}_X \to F$, the equality $\mathfrak{T} = \mathrm{h}_{\mathfrak{T}_X(\mathrm{id}_X)}$ holds. Hence, take $s \in \mathrm{h}_X(S)$ a morphism $s : S \to X$. Since $\mathfrak{T}$ is a natural transformation, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{h}_X(X) & \xrightarrow{\mathfrak{T}_X} & F(X) \\ \big\downarrow{\scriptstyle \mathrm{h}_X(s)} & & \big\downarrow{\scriptstyle s^*} \\ \mathrm{h}_X(S) & \xrightarrow{\mathfrak{T}_S} & F(S) \end{array}$$

Hence, applying it to $\mathrm{id}_X$, we have

$$\mathfrak{T}_S(s) = \mathfrak{T}_S(s \circ \mathrm{id}_X) = s^*\mathfrak{T}_X(\mathrm{id}_X) = \mathrm{h}_{\mathfrak{T}_X(\mathrm{id}_X)}(s)$$

and this means $\mathrm{h}_{\mathfrak{T}_X(\mathrm{id}_X)} = \mathfrak{T}$. □

## 1.4 Limits and colimits

Fix a category $\mathcal{C}$.

**Definition 1.16.** A *diagram* in $\mathcal{C}$ is a couple $(\mathcal{J}, \mathcal{F})$ were $\mathcal{J}$ is a category and $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ is a functor, $\mathcal{J}$ is called the *shape* of the diagram. The diagram is *finite* if $\mathcal{J}$ is a finite category.

**Definition 1.17.** A *cone* of the diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ is a couple $(N, \varphi)$ where $N$ is an object of $\mathcal{C}$ and $\varphi$ is a family of morphisms $\varphi_X : N \to \mathcal{F}(X)$ such that, if $g : X \to Y$ is a morphism in $J$, then $\varphi_Y = \mathcal{F}(g) \circ \varphi_X : N \to \mathcal{F}(X) \to \mathcal{F}(Y)$. A *morphism of cones* $(N, \varphi) \to (N', \varphi')$ is a morphism $f : N \to N'$ such that $\varphi_X = \varphi'_X \circ f$ for every object $X$ of $\mathcal{J}$.

$$
\begin{tikzcd}
 & N \arrow[dl] \arrow[dd] \arrow[dddl] \arrow[ddd, bend right] \arrow[dddr, bend left] & \\
\mathcal{F}(X_1) \arrow[dd] \arrow[ddr] & & \\
 & \mathcal{F}(X_2) \arrow[d] \arrow[dr] & \\
\mathcal{F}(X_3) \arrow[r] & \mathcal{F}(X_4) & \mathcal{F}(X_5)
\end{tikzcd}
$$

A *limit* of a diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ is an universal cone, i.e. a cone $(L, \psi)$ such that, for every cone $(N, \varphi)$, there exists a unique morphism of cones $(N, \varphi) \to (L, \psi)$. If a limit exists, then it is unique up to a unique isomorphism, and we will write it $\lim_{\mathcal{J}} \mathcal{F}$. It can be thought as a terminal object of the category of cones of $(\mathcal{J}, \mathcal{F})$. We will say that a category $\mathcal{C}$ admits limits of shape $\mathcal{J}$ if the limit in $\mathcal{C}$ exists for every diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$. Limits along diagrams of finite shape are called *finite limits*. Limits along diagrams of small shape are called *small limits*.

**Example 1.18.** Consider the category $\mathcal{J}$ of three elements and two morphism (not counting identities) shown in the following diagram:

$$
\begin{tikzcd}
 & X_2 \arrow[d] \\
X_1 \arrow[r] & X_3
\end{tikzcd}
$$

and let $\mathcal{F}$ be a functor from $\mathcal{J}$ to Sch. The limit of $\mathcal{F}$ always exists, and it is the fibered product $\mathcal{F}(X_1) \times_{\mathcal{F}(X_3)} \mathcal{F}(X_2)$.

$$
\begin{tikzcd}
N \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathcal{F}(X_1) \times_{\mathcal{F}(X_3)} \mathcal{F}(X_2) \arrow[r] \arrow[d] & \mathcal{F}(X_2) \arrow[d] \\
& \mathcal{F}(X_1) \arrow[r] & \mathcal{F}(X_3)
\end{tikzcd}
$$

In general, the limit of a diagram $(\mathcal{J}, \mathcal{F}')$ with $\mathcal{F}'$ a functor $\mathrm{Pr} \to \mathcal{C}$ is called *fibered product*.

The dual notions of cone and limit are co-cones and colimits.

**Definition 1.19.** A *co-cone* of the diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ is a couple $(N, \varphi)$ where $N$ is an object of $\mathcal{C}$ and $\varphi$ is a family of morphisms $\varphi_X : \mathcal{F}(X) \to N$ such that, if $g : X \to Y$ is a morphism in $J$, then $\varphi_X = \varphi_Y \circ \mathcal{F}(g) : \mathcal{F}(X) \to \mathcal{F}(Y) \to N$. A *morphism of co-cones* $(N, \varphi) \to (N', \varphi')$ is a morphism $f : N \to N'$ such that $\varphi'_X = f \circ \varphi_X$ for every object $X$ of $\mathcal{J}$.

$$
\begin{tikzcd}
\mathcal{F}(X_1) \arrow[dd] \arrow[ddr] \arrow[dddr, bend left=10] & & \\
& \mathcal{F}(X_2) \arrow[d] \arrow[dr] \arrow[ddd, bend left] & \\
\mathcal{F}(X_3) \arrow[r] \arrow[dr, bend right] & \mathcal{F}(X_4) \arrow[d] & \mathcal{F}(X_5) \arrow[dl, bend left] \\
& N &
\end{tikzcd}
$$

A *colimit* of a diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ is an universal co-cone, i.e. a co-cone $(L, \psi)$ such that, for every cone $(N, \varphi)$, there exists a unique morphism of co-cones $(L, \psi) \to (N, \varphi)$. If a colimit exists, then it is unique up to a unique isomorphism, and we will write it $\mathrm{colim}_{\mathcal{J}} \mathcal{F}$. It can be thought as an initial object of the category of co-cones of $\mathcal{F}$. We will say that a category $\mathcal{C}$ admits colimits of shape $\mathcal{J}$ if the limit in $\mathcal{C}$ exists for every diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$. Colimits along diagrams of finite shape are called *finite colimits*. Colimits along diagrams of small shape are called *small colimits*.

**Example 1.20.** Consider the category $\mathcal{J}$ of three elements and two morphism (not counting identities) shown in the following diagram:

$$\begin{tikzcd}
X_3 \arrow[r] \arrow[d] & X_1 \\
X_2 &
\end{tikzcd}$$

Now, let $\mathcal{F}$ be a functor from $\mathcal{J}$ to the category of commutative rings with identity. The colimit of $\mathcal{F}$ always exists and it is the tensor product $\mathcal{F}(X_1) \otimes_{\mathcal{F}(X_3)} \mathcal{F}(X_2)$.

$$\begin{tikzcd}
\mathcal{F}(X_3) \arrow[r] \arrow[d] & \mathcal{F}(X_1) \arrow[d] \arrow[ddr, bend left] & \\
\mathcal{F}(X_2) \arrow[r] \arrow[drr, bend right] & \mathcal{F}(X_1) \otimes_{\mathcal{F}(X_3)} \mathcal{F}(X_2) \arrow[dr, dashed, "\exists!"] & \\
& & N
\end{tikzcd}$$

In general, the limit of a diagram $(\mathcal{J}, \mathcal{F}')$ with $\mathcal{F}'$ a functor $\mathcal{J} \to \mathcal{C}$ is called *fibered coproduct*.

**Example 1.21.** Consider the category $\mathcal{J}$ of two elements and without morphisms except for identities.

Limits along $\mathcal{J}$ are simply products:

$$\begin{tikzcd}
N \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathcal{F}(X_1) \times \mathcal{F}(X_2) \arrow[r] \arrow[d] & \mathcal{F}(X_2) \\
& \mathcal{F}(X_1) &
\end{tikzcd}$$

Colimits along $\mathcal{J}$ are coproducts:

$$\begin{tikzcd}
& \mathcal{F}(X_1) \arrow[d] \arrow[ddr, bend left] & \\
\mathcal{F}(X_2) \arrow[r] \arrow[drr, bend right] & \mathcal{F}(X_1) \coprod \mathcal{F}(X_2) \arrow[dr, dashed, "\exists!"] & \\
& & N
\end{tikzcd}$$

**Proposition 1.22.** *Let $\mathcal{J}, \mathcal{K}$ be small categories and $\mathcal{F} : \mathcal{J} \times \mathcal{K} \to \mathcal{C}$ a diagram. For every object $j$ in $\mathcal{J}$ and $k$ in $\mathcal{K}$, consider the restrictions*

$$\mathcal{F}_j : \mathcal{K} \simeq \{j\} \times \mathcal{K} \to \mathcal{C}$$

*and*

$$\mathcal{F}_k : \mathcal{J} \simeq \mathcal{J} \times \{k\} \to \mathcal{C}.$$

*Suppose that $\mathcal{C}$ admits limits both of shape $\mathcal{J}$ and of shape $\mathcal{K}$. Call respectively $\lim_{\mathcal{J}} \mathcal{F} : \mathcal{K} \to \mathcal{C}$, $\lim_{\mathcal{K}} \mathcal{F} : \mathcal{J} \to \mathcal{C}$ the functors sending $k$ to $\lim_{\mathcal{J}} \mathcal{F}_k$ and $j$ to $\lim_{\mathcal{K}} \mathcal{F}_j$. Then, $\mathcal{C}$ admits limits of shape $\mathcal{J} \times \mathcal{K}$ and there are canonical isomorphisms*

$$\lim_{\mathcal{K}} \lim_{\mathcal{J}} \mathcal{F} \simeq \lim_{\mathcal{J} \times \mathcal{K}} \mathcal{F} \simeq \lim_{\mathcal{J}} \lim_{\mathcal{K}} \mathcal{F}.$$

*Roughly speaking, small limits commute with small limits.*

*Proof.* Since the problem is symmetric, it is enough to show that

$$\lim_{\mathcal{K}} \lim_{\mathcal{J}} \mathcal{F}$$

is a limit for $\mathcal{F}$.

Take an object $(j, k)$ in $\mathcal{J} \times \mathcal{K}$. By definition, there exist morphisms

$$\alpha_{j,k} : \lim_{\mathcal{K}} \lim_{\mathcal{J}} \mathcal{F} \to \lim_{\mathcal{J}} \mathcal{F}(k) = \lim_{\mathcal{J}} \mathcal{F}_k$$

and

$$\beta_{j,k} : \lim_{\mathcal{J}} \mathcal{F}_k \to \mathcal{F}_k(j) = \mathcal{F}(j, k),$$

their composition defines a morphism

$$\psi_{j,k} = \beta_{j,k} \circ \alpha_{j,k} : \lim_{\mathcal{K}} \lim_{\mathcal{J}} \mathcal{F} \to \mathcal{F}(j, k).$$

This makes $(\lim_{\mathcal{K}} \lim_{\mathcal{J}} \mathcal{F}, \psi)$ a cone of $\mathcal{F}$: we want to show that it is universal.

Hence, take $(N, \varphi)$ another cone of $\mathcal{F}$, we want to show that there exists a unique morphism $(N, \varphi) \to (\lim_{\mathcal{K}} \lim_{\mathcal{J}} \mathcal{F}, \psi)$. Let $k$ be an object in $\mathcal{K}$. By definition of $\lim_{\mathcal{J}} \mathcal{F}_k$, there exists a unique morphism $(N, \varphi|_{\mathcal{J}\times\{k\}}) \to (\lim_{\mathcal{J}} \mathcal{F}_k, \beta_k)$, and this in turn yields to a unique morphism $N \to \lim_{\mathcal{K}} \lim_{\mathcal{J}} \mathcal{F}$. The fact that this defines a morphism of cones is a simple check. $\square$

**Corollary 1.23.** *Small colimits commute with small colimits.*

*Proof.* If $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ is a diagram, $\operatorname{colim}_{\mathcal{J}} \mathcal{F} = \lim_{\mathcal{J}^{\mathrm{op}}} \mathcal{F}^{\mathrm{op}}$, hence we can apply Proposition 1.22. $\square$

**Corollary 1.24.** *Small limits commute with products, colimits of small shape commute with coproducts.* $\square$

**Proposition 1.25.** *Let $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ be a diagram, $\mathcal{G} : \mathcal{C} \to \mathcal{A}$ a functor and $\mathcal{H} : \mathcal{A} \to \mathcal{C}$ a left adjoint to $\mathcal{G}$. Then, if $\lim_{\mathcal{J}} \mathcal{F}$ exists, $\mathcal{G}(\lim_{\mathcal{J}} \mathcal{F})$ is a limit for $\mathcal{G} \circ \mathcal{F}$. Roughly speaking, if $\mathcal{G}$ has a left adjoint, then it commutes with limits. Dually, if $\mathcal{G}$ has a right adjoint, then it commutes with colimits.*

*Proof.* Let $(L, \varphi)$ be a universal cone for $\mathcal{F}$, we want to prove that $(\mathcal{G}L, \mathcal{G}\varphi)$ is a universal cone for $\mathcal{G} \circ \mathcal{F}$. Clearly, it is a cone thanks to the functoriality of $\mathcal{G}$, we want to see that it is universal.

For every $A$ in $\mathcal{A}$ and $C$ in $\mathcal{C}$, let

$$\theta_{A,C} : \operatorname{Hom}(\mathcal{H}A, C) \xrightarrow{\sim} \operatorname{Hom}(A, \mathcal{G}C)$$

be the bijection defining the adjunction and $(A, \psi)$, $(C, \eta)$ cones respectively for $\mathcal{G} \circ \mathcal{F}$ and $\mathcal{F}$. We have that

$$(\mathcal{H}A, \theta^{-1}\psi)$$

is a cone for $\mathcal{F}$ thanks to naturality of $\theta$: if $\sigma : j \to j'$ is a morphism in $\mathcal{J}$,

$$\theta^{-1}(\psi_{j'}) = \theta^{-1}(\mathcal{G}\mathcal{F}\sigma \circ \psi_j) = \mathcal{F}\sigma \circ \theta^{-1}(\psi_j).$$

Now, $\operatorname{Hom}((\mathcal{H}A, \theta^{-1}\psi), (C, \eta))$ is a subset of $\operatorname{Hom}(\mathcal{H}A, C)$ and $\operatorname{Hom}((A, \psi), (\mathcal{G}C, \mathcal{G}\eta))$ is a subset of $\operatorname{Hom}(A, \mathcal{G}C)$. We claim that $\theta$ respects these subsets.

In fact, a morphism $f : \mathcal{H}A \to C$ is a morphism of cones if and only if $\eta_j \circ f = \theta^{-1}(\psi_j)$ for every $j$, and $\theta(f)$ is a morphism of cones if and only if $\mathcal{G}\eta_j \circ \theta(f) = \psi_j$ for every $j$. But, thanks to naturality of $\theta$,

$$\theta(\eta_j \circ f) = \mathcal{G}\eta_j \circ \theta(f).$$

Now, the fact that $(L, \varphi)$ is a universal cone simply means that $\mathrm{Hom}((C, \eta), (L, \varphi))$ has exactly one element for every cone $(C, \eta)$. Hence $(\mathcal{G}L, \mathcal{G}\varphi)$ is universal, too, because

$$\mathrm{Hom}((A, \psi), (\mathcal{G}L, \mathcal{G}\varphi)) \simeq \mathrm{Hom}((\mathcal{H}A, \theta^{-1}\psi), (L, \psi))$$

has exactly one element for every cone $(A, \psi)$. $\square$

# Chapter 2

# Group-schemes

## 2.1 General theory

### 2.1.1 Definitions

For the rest of this chapter, we will fix a base scheme $S$: all schemes will be schemes over $S$ and all morphisms will be morphisms of schemes over $S$, all products without specified base scheme will be over $S$.

A group-scheme is a scheme $G$ with some additional structure that makes it similar to the usual notion of group. We can't simply put a regular operation on its points making it a group as we do, for example, for Lie groups, because morphisms between schemes involve structure sheaves. For example, there are a lot of morphism of $k$-schemes $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$, but set theoretically they are simply points, and there is only one function between them. The solution is to define them in terms of morphisms of schemes, as we would do with classical groups if we would want to define them in terms of functions between sets rather than in terms of operations on points.

In order to do this, we give three morphisms:

- A morphism $m : G \times G \to G$ called multiplication.
- A morphism $\varepsilon : S \to G$ called identity.
- A morphism $i : G \to G$ called inverse.

Moreover, these three morphisms must satisfy some constraints: the following diagrams must be commutative

- Associativity:

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id} \times m"] \arrow[d, "m \times \mathrm{id}"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

- Inverse:

$$\begin{tikzcd} G \arrow[r, "\mathrm{id} \times i"] \arrow[d] & G \times G \arrow[d, "m"] \\ S \arrow[r, "\varepsilon"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "i \times \mathrm{id}"] \arrow[d] & G \times G \arrow[d, "m"] \\ S \arrow[r, "\varepsilon"] & G \end{tikzcd}$$

- Identity:

$$\begin{tikzcd} S \times G \arrow[r, "\varepsilon \times \mathrm{id}"] \arrow[dr, "pr_2"'] & G \times G \arrow[d, "m"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} G \times S \arrow[r, "\mathrm{id} \times \varepsilon"] \arrow[dr, "pr_1"'] & G \times G \arrow[d, "m"] \\ & G \end{tikzcd}$$

If $S = \operatorname{Spec} R$ and $G = \operatorname{Spec} A$ are affine, it is useful to characterize the structure of group-scheme of $G$ in terms of the algebra $A$. Hence, we get three homomorphisms (with abuse of notation, indicated by the same letters):

- A homomorphism $m : A \to A \otimes_R A$ called comultiplication.
- A homomorphism $\varepsilon : A \to R$ called coidentity.
- A homomorphism $i : A \to A$ called coinverse.

Again, they must satisfy analogous constraints

- Coassociativity:

$$\begin{tikzcd} A \otimes_R A \otimes_R A & A \otimes_R A \arrow[l, "\mathrm{id} \otimes m"'] \\ A \otimes_R A \arrow[u, "m \otimes \mathrm{id}"] & A \arrow[l, "m"] \arrow[u, "m"'] \end{tikzcd}$$

- Coinverse:

$$\begin{tikzcd} A & A \otimes_R A \arrow[l, "\mathrm{id} \otimes i"'] \\ R \arrow[u] & A \arrow[l, "\varepsilon"] \arrow[u, "m"'] \end{tikzcd} \qquad \begin{tikzcd} A & A \otimes_R A \arrow[l, "i \otimes \mathrm{id}"'] \\ R \arrow[u] & A \arrow[l, "\varepsilon"] \arrow[u, "m"'] \end{tikzcd}$$

- Coidentity:

$$\begin{tikzcd} R \otimes_R A & A \otimes_R A \arrow[l, "\varepsilon \otimes \mathrm{id}"'] \\ & A \arrow[ul] \arrow[u, "m"'] \end{tikzcd} \qquad \begin{tikzcd} A \otimes_R R & A \otimes_R A \arrow[l, "\mathrm{id} \otimes \varepsilon"'] \\ & A \arrow[ul] \arrow[u, "m"'] \end{tikzcd}$$

Commutative algebras with this additional structure are called commutative Hopf algebras.

A morphism of group-schemes $G \to H$ is a morphism of schemes compatible with multiplication, i.e. making the following diagram commutative:

$$\begin{tikzcd} G \times G \arrow[r] \arrow[d] & H \times H \arrow[d] \\ G \arrow[r] & H \end{tikzcd}$$

This make group-schemes over $S$ a category.

From now on, we will always assume $S = \operatorname{Spec} k$ is the spectrum of a field.

**Definition 2.1.** We say that a morphism of affine group-schemes $G = \operatorname{Spec} A \to H = \operatorname{Spec} B$ is a *quotient* if $B \to A$ is injective. We will also say that $H$ is a quotient of $G$ tacitly supposing that is given a quotient morphism $G \to H$.

**Proposition 2.2.** *A quotient of group-schemes* $G = \operatorname{Spec} A \to H = \operatorname{Spec} B$ *is faithfully flat.*

*Proof.* This is proved in [Wat79, section 14]. $\square$

**Definition 2.3.** We will say that an homomorphism of group-schemes $G \to H$ is a *closed subgroup* if it is a closed immersion.

If $\varphi : G = \operatorname{Spec} A \to H = \operatorname{Spec} B$ is an homomorphism of affine groups, we may take $C = \varphi^{\#}(B) \subseteq A$ which is an Hopf subalgebra of $A$. Hence, $L = \operatorname{Spec} C \to H$ is a closed subgroup and $G \to L$ is a quotient. We call $L$ the *image* of $\varphi$. If $\varepsilon : \operatorname{Spec} k \to H$ is the identity, $K = G \times_H \operatorname{Spec} k$ is called the *kernel* of $\varphi$. The natural map $A \to A \otimes_B k$ is clearly surjective, hence $K \to G$ is a closed immersion. Moreover, the structure of group-scheme of $G$ naturally induces a structure of group-scheme on $K$.

### 2.1.2 Functorial point of view

Our definition of group-scheme tends to be rather cumbersome: you can't simply consider $G$ as a group over its points, you always need to work with morphisms instead of points. If you work on an algebraically closed field Nullstellensatz will help you, but here we are concerned with much more general objects. A way to overcome this lack of intuition is using the Yoneda Lemma to regard a group-scheme $G$ as a functor.

**Proposition 2.4.** *Giving to a scheme $G$ a structure of group-scheme is like defining a group structure on every set $G(U)$, where $U$ is a scheme, such that the maps $G(U) \to G(V)$ induced by morphisms $V \to U$ are group homomorphisms. Moreover, this is like asking the functor* $\mathrm{h}_G : \mathrm{Sch}/k \to \mathrm{Set}$ *to split as* $\mathrm{Sch}/k \to \mathrm{Grp} \to \mathrm{Set}$*, where* $\mathrm{Grp} \to \mathrm{Set}$ *is the forgetful functor.*

*Proof.* Take a scheme $U$, the multiplication $m : G \times G \to G$ induce, by the Yoneda Lemma, a multiplication $G(U) \times G(U) \to G(U)$. Similarly, we have an inverse $G(U) \to G(U)$ and an identity $S(U) = pt. \to G(U)$. The commutative diagrams of associativity, identity and inverse induce similar diagrams on $G(U)$, hence defining a group structure on $G(U)$. If we have a morphism $V \to U$, the induced map $G(U) \to G(V)$ is a group homomorphism because the functoriality of $G$ makes the following diagram commutative:

$$\begin{tikzcd} G(U) \times G(U) \arrow[r] \arrow[d] & G(U) \arrow[d] \\ G(V) \times G(V) \arrow[r] & G(V) \end{tikzcd}$$

On the other hand, the fact that all the induced maps $G(U) \to G(V)$ are group homomorphism tells us that all the diagram as the one above are commutative, hence the maps $G(U) \times G(U) \to G(U)$ define a natural transformation from $G \times G$ to $G$, and this in turn yields to a morphism $G \times G \to G$ thanks to the Yoneda Lemma. Similarly, one defines the inverse and the identity using the fact that $G(U) \to G(V)$ is a group homomorphism and the Yoneda Lemma. The fact that the diagrams of constraints are commutative descends from the analogous diagrams for the groups $G(U)$ using, again, the Yoneda Lemma.

It is now obvious that this is equivalent to splitting the functor $G$ as $\mathrm{Sch}/k \to \mathrm{Grp} \to \mathrm{Set}$, where $\mathrm{Grp} \to \mathrm{Set}$ is the forgetful functor. □

### 2.1.3 Examples

**Example 2.5.** Take a finite group $G$ and consider the disjoint union $\coprod_G \operatorname{Spec} k$. Since, for finite $G$, $\coprod_G \operatorname{Spec} k \times \coprod_G \operatorname{Spec} k \simeq \coprod_{G\times G} \operatorname{Spec} k$, there is an obvious way to put on $\coprod_G \operatorname{Spec} k$ the structure of a group-scheme. If a group-scheme is isomorphic to $\coprod_G S$ for some finite $G$, we call it a discrete group-scheme. This defines and embedding of the category of finite groups in the category of affine group-schemes over $k$.

**Example 2.6.** Take an integer $n$ and consider the set of $n$th roots of unity in the complex plane. They form a finite group, and we can take the discrete group associated to it. As a scheme, the set of $n$th roots of unity is $\mathbb{C}[x]/(x^n - 1)$, the comultiplication induced by the structure of discrete group is the homomorphism of algebras sending $x \mapsto x \otimes x$, the coidentity is simply $x \mapsto 1$ and the inverse is $x \mapsto x^{n-1}$. With this definition, it is clear that we don't really need $\mathbb{C}[x]/(x^n - 1)$: we can use $k[x]/(x^n - 1)$ with $k$ a generic field, define comultiplication, coidentity and coinverse in the same way and check that everything works. This, in general, will not be isomorphic to the discrete cyclic group of order $n$ over $\operatorname{Spec} k$, and it is called $\mu_n$.

**Example 2.7.** Consider the affine scheme $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ and a scheme $X$ over $k$. The elements $\mathbb{A}^1_k(X)$ are homomorphisms $k[x] \to \mathrm{H}^0(X)$ of $k$-algebras which are determined by the image of $x$: as sets we may identify $\mathbb{A}^1_k(X) = \mathrm{H}^0(X)$. Hence, on $\mathbb{A}^1_k(X)$ we have a natural structure of group given by sum on $\mathrm{H}^0(X)$. This defines an affine group-scheme $\mathbb{G}_a$ on the affine line $\mathbb{A}^1_k$, called the additive group.

Similarly, fix a vector space $V$ and consider the $k$-algebra

$$A = \operatorname{Sym}(V^\vee).$$

If $X$ is a scheme over $k$, the $X$-points of $\operatorname{Spec} A$ are homomorphisms $\operatorname{Sym}(V^\vee) \to \mathrm{H}^0(X)$, corresponding to $k$-linear maps $V^\vee \to \mathrm{H}^0(X)$. When $\dim V < +\infty$, we may regard $\operatorname{Spec} A$ as the functor sending $X$ to $\mathrm{H}^0(X) \otimes V$. We have an obvious structure of group on $V \otimes \mathrm{H}^0(X)$, defining a structure of group-scheme on $\operatorname{Spec} A$ thanks to the Yoneda Lemma.

We will call $V_{\mathrm{sch}} = \operatorname{Spec} A$. If there is no risk of confusion, we will call both $V$ both the vector space and the scheme.

Now consider $\mathbb{A}^1_k \setminus \{0\} = \operatorname{Spec} k[x]_x$. As before, the elements of $\mathbb{A}^1_k \setminus \{0\}(X)$ are homomorphisms $k[x]_x \to \mathrm{H}^0(X)$, and this identifies $\mathbb{A}^1_k \setminus \{0\}(\mathrm{H}^0(X))$ with $\mathrm{H}^0(X)^*$ which has a natural structure of group with multiplication. This defines an affine group-scheme $\mathbb{G}_m$ on $\operatorname{Spec} k[x]_x$ called the multiplicative group of $k$.

**Example 2.8.** Fix a finite dimensional vector space $V$ and consider the $k$-module $\operatorname{Hom}(V, V)$ of linear maps $V \to V$. We may identify $\operatorname{Hom}(V, V)$ with the rational points of $\operatorname{Spec} A$, where $A = \operatorname{Sym}(V^\vee \otimes V)$. In general, fix a scheme $\mathrm{H}^0(X)$ and consider the set of $X$-points $\operatorname{Spec} A(X)$. A point $p \in \operatorname{Spec} A(X)$ is an homomorphism of rings $A \to \mathrm{H}^0(X)$, which in turn corresponds to a $k$-linear map $V^\vee \otimes V \to \mathrm{H}^0(X)$. Hence, $\operatorname{Spec} A(X)$ may be identified with $\mathrm{H}^0(X)$-linear maps $V \otimes \mathrm{H}^0(X) \to V \otimes \mathrm{H}^0(X)$. As a functor, we may regard $\operatorname{Spec} A$ as $X \mapsto V \otimes \mathrm{H}^0(X)$.

Now, fix a basis of $V$ and use it to define the determinant as an element $\det \in A$. If $\varphi : A \to \mathrm{H}^0(X)$ is a point of $\operatorname{Spec} A(X)$, $\varphi(\det)$ is the determinant of the corresponding $\mathrm{H}^0(X)$-linear map $V \otimes \mathrm{H}^0(X) \to V \otimes \mathrm{H}^0(X)$, hence the spectrum of $A_{\det}$ is an open subscheme of $\operatorname{Spec} A$ representing the group of $\mathrm{H}^0(X)$-linear automorphisms of $V \otimes \mathrm{H}^0(X)$. This defines a structure of group-scheme on $\operatorname{Spec} A_{\det}$ thanks to the Yoneda Lemma. We will call $\operatorname{Hom}(V, V)$, $\mathrm{GL}(V)$ both the schemes $\operatorname{Spec} A$, $\operatorname{Spec} A_{\det}$ and the classical objects. If the situation is ambiguous, we will specify $\operatorname{Hom}(V, V)(k)$ and $\mathrm{GL}(V)(k)$ for the classical objects. We will also shorten $\mathrm{GL}(k^n)$ as $\mathrm{GL}_n$.

Finally, one may find all the usual matrix groups as closed subgroups of $\mathrm{GL}_n$: $\mathrm{SL}_n$, $\mathrm{O}_n$, $\mathrm{SO}_n$ and so on.

## 2.2 Connected components

In a topological group $G$, the connected component $G^\circ$ of the identity is a normal subgroup, and the quotient $\pi_0(G) = G/G^\circ$ is the set of connected components of $G$. An analogous statement is true for group-schemes of finite type over a field, as we shall see in this section.

### 2.2.1 Étale algebras

We are going study étale $k$-algebras from an abstract point of view. The reason why we are interested in them is that they are the right tool to extend classical results about connectedness. Classically, one may describe the set of connected components of a variety $X$ as the largest finite set $S$ with a surjective map $X \to S$. We want to do exactly the same thing, but taking into account the underlying algebraic structure. This will lead us to define the étale scheme $\pi_0(\operatorname{Spec} A)$ as the maximal étale affine scheme $\operatorname{Spec} B$ with a dominant map $\operatorname{Spec} A \to \operatorname{Spec} B$. In fact, when $k = \bar{k}$, Theorem 2.20 says that the category of étale affine schemes over $k$ is equivalent to the category of finite sets.

**Definition 2.9.** Let $f : X \to Y$ be a morphism locally of finite presentation. Let $x \in X$ and $y = f(x)$. We say that $f$ is *unramified at* $x$ if $\mathfrak{m}_y \mathcal{O}_{X,x} = \mathfrak{m}_x$ (in other words, if $\mathcal{O}_{X_y,x} = k(x)$) and $k(x)/k(y)$ is separable. We say that $f$ is *unramified* if it is unramified at all points of $X$.

**Lemma 2.10.** *Let $F/k$ be a finite extension of degree $n$.*

- *If $F/k$ is separable, $F \otimes K$ is a finite product of separable extensions of $K$.*
- *If $F \otimes \bar{k}$ is reduced, $F/k$ is separable.*

*Proof.* Let $F/k$ be separable. Thanks to the primitive element theorem, there exists $\alpha \in F$ such that $F = k(\alpha) \simeq k[x]/(p)$, with $p \in k[x]$ minimal polynomial of $\alpha$. In $K[x]$, since $p$ is separable it splits as $p(x) = \prod_{i=1}^{n} p_i(x)$ with $p_i \in K[x]$ separable irreducible and $p_i \neq p_j$ if $i \neq j$. Hence, thanks to the chinese remainder theorem,

$$F \otimes K \simeq K[x]/(p) \simeq$$

$$\simeq K[x]/(p_1) \cdots \times K[x]/(p_n) \simeq K_1 \times \cdots \times K_n.$$

On the other hand, suppose $F/k$ is not separable, and choose $\alpha \in F$ not separable over $k$. Since $k(\alpha) \otimes \bar{k} \subseteq F \otimes \bar{k}$, it is enough to show that $k(\alpha) \otimes \bar{k}$ is not reduced. Consider $p(x) \in k[x]$ the minimal polynomial of $\alpha$: since $\alpha$ is not separable, there exists $q > 1$ and a polynomial $p_1 \in \bar{k}[x]$ such that $p(x) = (x - \alpha)^q \cdot p_1(x)$, with $(x - \alpha) \nmid p_1(x)$. Hence,

$$k(\alpha) \otimes \bar{k} \simeq \bar{k}[x]/(p) \simeq \bar{k}[x]/(x-\alpha)^q \times \bar{k}[x]/(p_1)$$

is not reduced. □

**Lemma 2.11.** *Let $f : X \to Y$ be a morphism locally of finite presentation. Then $f$ is unramified if and only if, for every field $K$ and every morphism* $\operatorname{Spec} K \to Y$, *$X \times_Y$* $\operatorname{Spec} K$ *has the discrete topology and is reduced.*

*Proof.* Let us suppose that $f$ is unramified, and consider a morphism $\operatorname{Spec} K \to Y$ on a point $y \in Y$. Firstly, we will prove that $X_y$ has the discrete topology and is reduced, and then we will extend the result to $X \times_Y \operatorname{Spec} K$.

The problem is local, we may suppose $X_y = \operatorname{Spec} A$, with $A$ a $k(y)$-algebra of finite presentation, hence noetherian. Since for every prime $\mathfrak{p} \subseteq A$ the localization $A_{\mathfrak{p}}$ is a field, $A$ is a product of separable fields over $k(y)$. Then, thanks to Lemma 2.10, $X \times_Y \operatorname{Spec} K = \operatorname{Spec} A \otimes_{k(y)} K$ is a finite product of separable field extensions of $K$.

On the other hand, suppose that $X \times_Y \operatorname{Spec} K$ has the discrete topology and is reduced for every morphism $\operatorname{Spec} K \to Y$. Take a point $x \in X$ with $y = f(x)$, as before we may suppose $X_y = \operatorname{Spec} A$. Since $\operatorname{Spec} A$ has the discrete topology and is reduced and $A$ is noetherian, $A$ is a finite product of field extension of $k(y)$. We only need to show that these field extensions are separable, and this comes from the fact that $X_y \times_{\operatorname{Spec} k(y)} \operatorname{Spec} \bar{k(y)} = \operatorname{Spec} A \otimes_{k(y)} \bar{k(y)}$ is reduced, too, and from Lemma 2.10. □

**Corollary 2.12.** *Unramified morphisms are stable under base change.*

*Proof.* The condition of Lemma 2.11 is obviously stable under base change. □

**Definition 2.13.** A morphism $f : X \to Y$ is *étale* if it is both flat and unramified.

**Corollary 2.14.** *Étale morphisms are stable under base change.*

*Proof.* Both flat and unramified morphisms are stable under base change. □

Let $A$ be an étale algebra over $k$ (i.e. $\operatorname{Spec} A \to \operatorname{Spec} k$ is étale). We know that $A$ is finite and reduced over $k$, hence $A = \prod_{i=1}^{n} k_i$. Moreover, $k_i/k$ is a finite separable extension. On the other hand, every algebra of the form $\prod_{i=1}^{n} k_i$ with $k_i/k$ separable is clearly étale over $k$. This gives us a simple characterization of étale $k$-algebras.

Now, let $k_s$ be a separable closure of $k$: if $k = k_s$, étale $k$-algebras are simply finite products of copies of $k$. If $A$ is a finite product of copies of $k$, we will say that $A$ is diagonalizable.

**Proposition 2.15.** *A $k$-algebra $A$ is étale if and only if $A \otimes k_s$ is diagonalizable over $k$.*

*Proof.* If $F/k$ is a separable extension, $F \otimes k_s \simeq {k_s}^n$ where $n$ is the degree of $F/k$ thanks to Lemma 2.10, hence $A \otimes k_s$ is diagonalizable when $A$ is étale.

On the other hand, suppose $A \otimes k_s$ diagonalizable. Since the homomorphism $A \to A \otimes k_s$ sending $a$ to $a \otimes 1$ is injective, $A$ is reduced. Moreover, since $\dim_k(A) = \dim_{k_s}(A \otimes k_s)$ as vector spaces, $A$ is finite, hence it is a finite product of local, reduced and artinian rings, i.e. fields.

Let $F$ be a field contained in $A$. Since $A \otimes \bar{k} = (A \otimes k_s) \otimes_{k_s} \bar{k}$ is diagonalizable, $F \otimes \bar{k} \subseteq A \otimes \bar{k}$ is reduced and hence $F/k$ is separable thanks to Lemma 2.10. □

**Corollary 2.16.** *Subalgebras, quotients, finite products and tensor products of étale k-algebras are étale.*

*Proof.* The statement is obvious when $k$ is separably closed, because in this case étale simply means diagonalizable. The general case is implied by the particular one simply applying $\cdot \otimes k_s$ everywhere. □

**Corollary 2.17.** *If $B, C$ are étale subalgebras of a k-algebra $A$, the composite $BC \subseteq A$ is étale.*

*Proof.* The subalgebra $BC$ is the image of the map $B \otimes C \to A$ sending $b \otimes c$ to $bc$. □

Discrete group-schemes, as defined in Example 2.5, are finite, but in general they are not the only ones. For example, if $\operatorname{char} k = p$, the group-scheme $\mu_p = \operatorname{Spec} k[x]/(x^p - 1)$ is finite but not reduced. This is not the only problem: a finite group-scheme may be étale, but still not discrete. If $k = \mathbb{R}$, the group-scheme $\mu_3$ is $\operatorname{Spec} \mathbb{R} \oplus \mathbb{C}$, hence it is étale but not discrete. Still, when $\operatorname{char} k = 0$ and $k = \bar{k}$, the situation is closer to classical intuition:

**Theorem 2.18** (Cartier). *If* $\operatorname{char} k = 0$, *finite group-schemes are étale.*

*Proof.* This is proved in [Wat79, sect. 11.4]. □

**Corollary 2.19.** *If* $\operatorname{char} k = 0$ *and* $k = \bar{k}$, *finite group-schemes are discrete.* □

Now take an étale algebra $A$ and consider the set of $k_s$ rational points of $\operatorname{Spec} A$, it is $\operatorname{Hom}(A, k_s)$. Let $\Gamma$ be the Galois group $\operatorname{Gal}(k_s/k)$, $\Gamma$ acts on $\operatorname{Spec} A(k_s)$ with its action on $k_s$. Since $A$ is finite over $k$, there exist a Galois extension $L/k$ such that the images of all homomorphisms $A \to k_s$ are contained in $L$. This means that the action of $\Gamma$ on $\operatorname{Spec} A(k_s)$ factors through its finite quotient $\operatorname{Gal}(L/k)$, hence the action is continuous. Moreover, an homomorphism $f : A \to B$ between étale algebras induces a $\Gamma$ equivariant map $\operatorname{Spec} B(k_{\mathrm{s}}) \to \operatorname{Spec} A(k_{\mathrm{s}})$, defining a contravariant functor $\mathcal{F}$ from the category of étale $k$-algebras to the category of finite sets with a continuous action of $\Gamma$.

**Theorem 2.20.** *The functor $\mathcal{F}$ sending $A$ to* $\operatorname{Spec} A(k_{\mathrm{s}})$ *establishes an equivalence between the opposite category of étale k-algebras and finite sets with a continuous action of* $\operatorname{Gal}(k_s/k)$*.*

*Proof.* We begin by showing that $\mathcal{F}$ is fully faithful.

Let $S$ be a set with a continuous action of $\Gamma$. The set $k_{\mathrm{s}}^S$ of functions $f : S \to k_{\mathrm{s}}$ is an étale $k_{\mathrm{s}}$-algebra with $\gamma \in \Gamma$ acting on $k_{\mathrm{s}}^S$ by sending $f : S \to k_{\mathrm{s}}$ to $\gamma \circ f \circ \gamma^{-1}$.

Now, let $A$ be an étale $k$-algebra. If we let $\Gamma$ act on $A \otimes k_{\mathrm{s}}$ with its action on $k_{\mathrm{s}}$, the map $A \otimes k_{\mathrm{s}} \to k_{\mathrm{s}}^{\mathcal{F}(A)}$ sending

$$a \otimes c \mapsto (\sigma \mapsto c\sigma(a))$$

is a $\Gamma$-equivariant isomorphism. Hence,

$$A \simeq (A \otimes k_{\mathrm{s}})^{\Gamma} \simeq (k_{\mathrm{s}}^{\mathcal{F}(A)})^{\Gamma}.$$

This shows that $\mathcal{F}$ is faithful. Now, to show that $\mathcal{F}$ is full, let $A$ and $B$ be étale algebras and $\varphi : \mathcal{F}(A) \to \mathcal{F}(B)$ a map. Consider the induced function $\varphi^* : (k_{\mathrm{s}}^{\mathcal{F}(B)})^{\Gamma} \to (k_{\mathrm{s}}^{\mathcal{F}(A)})^{\Gamma}$: we claim that $\mathcal{F}(\varphi^*) = \varphi$. Take $a \in \mathcal{F}(A)$, as a point of $\mathcal{F}(A) \simeq \mathcal{F}((k_{\mathrm{s}}^{\mathcal{F}(A)})^{\Gamma}) = (k_{\mathrm{s}}^{\mathcal{F}(A)})^{\Gamma}(k_{\mathrm{s}})$ it is $f \mapsto f(a)$, where $f \in (k_{\mathrm{s}}^{\mathcal{F}(A)})^{\Gamma}$. Hence, $\mathcal{F}(\varphi^*)(a)$ correspond to $g \mapsto \varphi^*(g)(a) = g(\varphi(a))$ in $\mathcal{F}(B) \simeq \mathcal{F}((k_{\mathrm{s}}^{\mathcal{F}(B)})^{\Gamma})$, with $g \in (k_{\mathrm{s}}^{\mathcal{F}(A)})^{\Gamma}$. Hence, $\mathcal{F}(\varphi^*)(a) = \varphi(a)$.

Now we want to prove that $\mathcal{F}$ is essentially surjective. Let $S$ be a set with a continuous action of $\Gamma$. Since $\mathcal{F}(A \times B) = \mathcal{F}(A) \sqcup \mathcal{F}(B)$, we may suppose that the action of $\Gamma$ on $S$ is transitive.

Since the action is continuous, there exists a finite Galois extension $L/k$ such that the action of $\Gamma$ factors through $\operatorname{Gal}(L/k)$. Now fix a point $i \in S$, and consider the subfield $A \subseteq L$ fixed by $\operatorname{Stab}_{\operatorname{Gal}(L/k)}(i)$. We claim that $\mathcal{F}(A) \simeq S$. In fact, call $x_0 \in A(k_{\mathrm{s}})$ the point corresponding to the inclusion $A \subseteq L \subseteq k_{\mathrm{s}}$. Since the action of $\operatorname{Gal}(L/k)$ on $A(k_{\mathrm{s}})$ is obviously transitive, if we show that $\operatorname{Stab}_{\operatorname{Gal}(L/k)}(i) = \operatorname{Stab}_{\operatorname{Gal}(L/k)}(x_0)$ we have finished, and this is exactly Galois correspondence between subgroups of $\operatorname{Gal}(L/k)$ and subfields of $L$.

$\square$

**Corollary 2.21.** *If $\Gamma = \mathrm{Gal}(k_s/k)$ acts continuously on a finite set $S$, the natural map $\psi : (k_s^S)^\Gamma \otimes k_s \to k_s^S$ is a $\Gamma$-equivariant isomorphism.*

*Proof.* Let $A$ be an étale algebra such that $\mathcal{F}(\mathcal{A}) \simeq S$, we have already seen that
$$A \simeq (k_s^{\mathcal{F}(\mathrm{A})})^\Gamma \simeq (k_s^{\mathrm{S}})^\Gamma.$$

Using this isomorphism one may check that $\psi$ corresponds to the map $A \otimes k_s \to k_s^{\mathcal{F}(A)}$ defined in Theorem 2.20, and we already know that this is a $\Gamma$-equivariant isomorphism. □

### 2.2.2 The subalgebra of connected components

**Proposition 2.22.** *Let $A$ be a $k$-algebra of finite type. There exists an étale subalgebra $\pi_0(A) \subseteq A$ containing all the étale subalgebras of $A$.*

*Proof.* Let $B \subseteq A$ be an étale subalgebra, $B \otimes k_s = {k_s}^n$ where $n = \dim_k B$. Let $e_i$ be $(0, \ldots, 0, 1, 0, \ldots, 0) \in B \otimes k_s$, where 1 is in the $i$-th place. Since $e_1 + \cdots + e_n = 1$ and $e_i e_j = 0$ if $i \neq j$, we have a decomposition $\operatorname{Spec} A \otimes k_s = D(e_1) \sqcup D(e_2) \sqcup \cdots \sqcup D(e_n)$ with $D(e_i)$ open and closed. Hence, $n$ is limited by the number of connected components of $\operatorname{Spec} A \otimes k_s$, which is finite because $A \otimes k_s$ is noetherian.

Since the degree of an étale subalgebra of $A$ is limited, the thesis descends immediately from the fact that finite compositions of étale subalgebras are étale, as proved in Corollary 2.17. □

**Lemma 2.23.** *If $A$ is a $k$-algebra of finite type, $\pi_0(A) \otimes \bar{k} = \pi_0(A \otimes \bar{k})$.*

*Proof.* We start by proving the analogous statement with $k_s$ instead of $\bar{k}$.

Clearly, $\pi_0(A) \otimes k_s$ is étale, and hence we have
$$\pi_0(A) \otimes k_s \subseteq \pi_0(A \otimes k_s).$$

Now, $\Gamma = \mathrm{Gal}(k_s/k)$ acts on $A \otimes k_s$ with the action on $k_s$. Let $B$ be an étale $k_s$-subalgebra of $A \otimes k_s$ stable under the action of $\Gamma$. Let $\{\sigma_i\}_{i \in S} = \mathrm{Hom}_{k_s\mathrm{alg}}(B, k_s)$ be the set of $k_s$-points of $\operatorname{Spec} B$ indexed by a set $S$, with $\gamma \in \Gamma$ acting on $S$ by sending $i$ to the only $j$ such that
$$\sigma_j = \gamma \circ \sigma_i \circ \gamma^{-1} : B \to B \to k_s \to k_s.$$

Let us suppose for a moment that the action on $S$ is continuous. Note that this is *not* obvious: we have seen that for an étale $k$-algebra, the action on $k_s$ rational points is continuous, now we have an étale $k_s$-algebra with an action of $\Gamma$: we must somehow use that $B$ is a subalgebra of a $A \otimes k_s$.

Since $b \mapsto (i \mapsto \sigma_i(b))$ defines a $\Gamma$-equivariant isomorphism $B \simeq k_s^S$ and the action on $S$ is continuous, thanks to Corollary 2.21 $B = B^\Gamma \otimes k_s$. In particular,

$$\pi_0(A \otimes k_s) = \pi_0(A \otimes k_s)^\Gamma \otimes k_s \subseteq \pi_0(A) \otimes k_s.$$

Hence, we are left with proving that the action on $S$ is continuous. We may write $B = \bigoplus_{i \in S} k_s e_i$ with $e_i$ idempotent, $e_i e_j = 0$ if $i \neq j$ and $\sigma_i(e_j) = \delta_{ij}$. Hence, for every $b \in B$, we have $b = \sum_{i \in S} \sigma_i(b) e_i$. In particular, if $\gamma \in \Gamma$

$$\gamma(e_j) = \sum_{i \in S} \sigma_i(\gamma(e_j)) e_i = \sum_{i \in S} \gamma \circ \gamma^{-1} \circ \sigma_i(\gamma(e_j)) e_i =$$

$$= \sum_{i \in S} \gamma(\sigma_{\gamma^{-1} i}(e_j)) e_i = \gamma(1) e_{\gamma j} = e_{\gamma j}.$$

This means that $\Gamma$ permutes the set of idempotents $\{e_i\}_{i \in S}$ coherently with the action on $S$. But the action on the set $\{e_i\}_{i \in S}$ is continuous: there exists a Galois subextension $L/k$ of $k_s/k$ such that every $e_i$ is contained in $A \otimes L$, and hence the action of $\mathrm{Gal}(k_s/k)$ on $\{e_i\}_{i \in S}$ factors through $\mathrm{Gal}(L/k)$.

To pass from $k_s$ to $\bar{k}$, take a set $\{\sum_j a_{i,j} \otimes c_{i,j}\}_{1 \leq i \leq n}$ of orthogonal idempotents in $A \otimes \bar{k}$, they describe an étale subalgebra $\bar{k}(\sum_j a_{1,j} \otimes c_{1,j}) \times \cdots \times \bar{k}(\sum_j a_{n,j} \otimes c_{n,j})$ of $A \otimes \bar{k}$. If $\bar{k} \neq k_s$, $\bar{k}/k_s$ is purely inseparable and there exists a prime $p$ and a nonnegative integer $r$ such that $c_{i,j}^{p^r} \in k_s$ for every $i, j$. But then,

$$\sum_j a_{i,j} \otimes c_{i,j} = \left( \sum_j a_{i,j} \otimes c_{i,j} \right)^{p^r} = \sum_j a_{i,j}^{p^r} \otimes c_{i,j}^{p^r} \in A \otimes k_s.$$

Hence, $\dim_{\bar{k}}(\pi_0(A \otimes \bar{k})) \leq \dim_{k_s}(\pi_0(A \otimes k_s))$, and this implies that the following injection is also surjective:

$$\pi_0(A \otimes k_s) \otimes_{k_s} \bar{k} \to \pi_0(A \otimes \bar{k}).$$

□

**Corollary 2.24.** *If $A$ is a $k$-algebra of finite type, the number of connected components of* $\mathrm{Spec}\, A \otimes \bar{k}$ *is* $\dim_k \pi_0(A)$.

*Proof.* Take a decomposition of $\mathrm{Spec}\, A \otimes \bar{k}$ as $\mathrm{Spec}\, A_1 \sqcup \cdots \sqcup \mathrm{Spec}\, A_n$, we have $A \otimes \bar{k} = A_1 \times \cdots \times A_n$. Let $e_i$ be the unit of $A_i$, $e_i$ is idempotent and $e_i e_j = 0$ if $i \neq j$. The subalgebra $\bar{k} e_1 \times \cdots \times \bar{k} e_n \subseteq A \otimes \bar{k}$ is étale.

On the other hand, if $\bar{k}e'_1 \times \cdots \times \bar{k}e'_m \subseteq A \otimes \bar{k}$ is an étale subalgebra, $A \otimes \bar{k} = Ae'_1 \times \cdots \times Ae'_m$ gives a decomposition of $\operatorname{Spec} A$ in open, disjoint subschemes.

Hence, the maximal étale subalgebra $\pi_0(A \otimes \bar{k})$ corresponds to the maximal decomposition of $\operatorname{Spec} A \otimes \bar{k}$: this implies that the number of connected components of $\operatorname{Spec} A$ is

$$\dim_{\bar{k}} \pi_0(A \otimes \bar{k}) = \dim_{\bar{k}} \pi_0(A) \otimes \bar{k} = \dim_k \pi_0(A).$$

□

**Proposition 2.25.** *If $A$, $A'$ are $k$-algebras of finite type, the natural map $\pi_0(A) \otimes \pi_0(A') \to \pi_0(A \otimes A')$ is an isomorphism.*

*Proof.* Thanks to Lemma 2.23, we may apply $\cdot \otimes \bar{k}$ and suppose $k = \bar{k}$.

We already know that $\pi_0(A) \otimes \pi_0(A') \to \pi_0(A \otimes A')$ is injective. Since $\dim_k(\pi_0(A))$ is the number of connected components of $\operatorname{Spec} A$, it is enough to prove that the number of connected components of $\operatorname{Spec} A \otimes A'$ is the product of the number of connected components of $\operatorname{Spec} A$ and $\operatorname{Spec} A'$. Thanks to [Bou64, V.3.4, Theorem 3], $A$, $A'$ and $A \otimes A'$ are Jacobson, and hence we may use Specm instead of Spec to study connected components. Thanks to Nullstellensatz, topologically:

$$\operatorname{Specm} A \otimes A' = \operatorname{Spec} A \otimes A'(k) =$$

$$= \operatorname{Spec} A(k) \times \operatorname{Spec} A'(k) = \operatorname{Specm} A \times \operatorname{Specm} A'.$$

□

**Corollary 2.26.** *If $A$ is an Hopf algebra of finite type, $\pi_0(A) \subseteq A$ is an Hopf subalgebra.*

*Proof.* Let $\rho : A \to A \otimes A$ be the comultiplication. Since $\pi_0(A \otimes A) = \pi_0(A) \otimes \pi_0(A)$, it is enough to show that $\rho(\pi_0(A))$ is étale. But the image of an étale $k$-algebra is an étale $k$-algebra, too, thanks to Corollary 2.16. □

If $G = \operatorname{Spec} A$ is a group-scheme of finite type, we call $\pi_0(G) = \operatorname{Spec} \pi_0(A)$. It is the maximal étale group-scheme $H$ with a quotient map $G \to H$. The kernel $G^\circ$ of $G \to \pi_0(G)$ is simply the connected component of the identity. In fact, write $\pi_0(A)$ as $k_0 \times B$, where $B$ is the kernel of the identity $\pi_0(A) \to k$ and $k_0 \simeq k$. We have that $G^\circ$ is simply $\operatorname{Spec} A \otimes_{k_0 \times B} k = \operatorname{Spec} k_0 A$, which is exactly the connected component of the identity.

**Proposition 2.27.** *Let $G = \operatorname{Spec} A$ be an affine group-scheme of finite type. The following are equivalent:*

*(a) $G$ is irreducible.*

*(b) $G$ is connected.*

*(c) $\pi_0(G)$ is trivial.*

*(d) $\pi_0(G \times \operatorname{Spec} \bar{k})$ is trivial.*

*(e) $G$ is geometrically connected.*

*(f) $G$ is geometrically irreducible.*

*Proof.* The implication $(a) \Rightarrow (b)$ is obvious.

For $(b) \Rightarrow (c)$, if $G$ is connected $\pi_0(G)$ is connected, too, and the existence of the identity $\operatorname{Spec} k \to \pi_0(G)$ implies that the only point of $\pi_0(G)$ is $k$-rational.

$(c) \Rightarrow (d)$ descends immediately from Lemma 2.23.

$(d) \Rightarrow (e)$ descends from Corollary 2.24.

For $(e) \Rightarrow (f)$, we note that since $A \otimes \bar{k}$ is Jacobson [Bou64, V.3.4, Theorem 3], it is enough to show that $G \times \operatorname{Spec} \bar{k}(\bar{k}) = \operatorname{Specm} G \times \operatorname{Spec} \bar{k}$ is irreducible. Since $G \times \operatorname{Spec} \bar{k}(\bar{k})$ is connected ($G \times \operatorname{Spec} \bar{k}$ is connected and Jacobson), if it is not irreducible we may find two different irreducible components $V, W$ such that $V \cap W$ is nonempty. Let $p \in V \cap W$ be a point in the intersection, since multiplication by $p^{-1}$ is an homeomorphism of $G \times \operatorname{Spec} \bar{k}(\bar{k})$ with itself we may suppose that the identity $e$ is in $V \cap W$. Now take the multiplication map $V \times W \to G \times \operatorname{Spec} \bar{k}(\bar{k})$: since $V \times W$ is irreducible, its image is irreducible, too, and contains both $V$ and $W$, absurd.

Finally, $(f) \Rightarrow (a)$ comes from the fact that the projection $G \times \operatorname{Spec} \bar{k} \to G$ is integral, hence surjective. □

## 2.3 Representations

### 2.3.1 Definitions

Given a functor $X : \mathrm{Sch}/k^{\mathrm{op}} \to \mathrm{Set}$, we can define an action of a group-scheme $G$ on $X$ as a morphism of functors $j : G \times X \to X$ making the diagrams

$$\begin{array}{ccc} \operatorname{Spec} k \times X & \xrightarrow{\varepsilon\times\mathrm{id}} & G \times X \\ \| & & \downarrow j \\ X & = & X \end{array} \qquad \begin{array}{ccc} G \times G \times X & \xrightarrow{m\times\mathrm{id}} & G \times X \\ \downarrow \mathrm{id}\times j & & \downarrow j \\ G \times X & \xrightarrow{j} & X \end{array}$$

commute.

This is simply an action of the group $G(U)$ on the set $X(U)$ for every scheme $U$, such that these actions are intertwined by the functions $X(U) \to X(V)$ and $G(U) \to G(V)$ induced by morphisms $V \to U$.

*Remark* 2.28. We have used almost nothing: we could simply define on an arbitrary category $\mathcal{C}$ a group functor $G : \mathcal{C}^{\mathrm{op}} \to \mathrm{Grp} \to \mathrm{Set}$ and a functor $X : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, and define an action as a natural transformation $G \times X \to X$ making $G(U) \times X(U) \to X(U)$ an action for every $U \in \operatorname{Obj} \mathcal{C}$.

**Example 2.29.** Multiplication $m : G \times G \to G$ obviously defines an action of $G$ on itself. We can also define, as in the classical case, an action of conjugation, simply defining the conjugation $G(S) \times G(S) \to G(S)$ for every scheme $S$ and using the Yoneda Lemma.

A particular type of action is a representation. Fix a vector space $V$ and consider the functor $X \mapsto V \otimes \mathrm{H}^0(X)$ from $\mathrm{Sch}/k^{\mathrm{op}}$ to Set: we will call it $V$ with abuse of notation. When $V$ is finite dimensional, we have seen in Example 2.7 that the functor is represented by the scheme $V_{\mathrm{sch}}$.

**Definition 2.30.** A representation of a group-scheme $G$ on $V$ is an action $G \times V \to V$ such that for every scheme $X$ the induced action

$$G(X) \times (V \otimes \mathrm{H}^0(X)) \to V \otimes \mathrm{H}^0(X)$$

is $\mathrm{H}^0(X)$-linear.

Every point of $G(X)$ defines a linear automorphism of $V \otimes \mathrm{H}^0(X)$, which is a point of $\mathrm{GL}(V)(X)$: this implies that giving a representation of $G$ on $V$ is like giving an homomorphism of group-schemes $G \to \mathrm{GL}(V)$.

### 2.3.2 Comodules

If $G = \operatorname{Spec} A$ is affine, we have in particular that $G(A)$ acts linearly on $V \otimes A$. Now take $\mathrm{id}_G \in G(A) = \operatorname{Hom}(G, G)$, it defines an $A$-linear function $V \otimes A \to V \otimes A$ that restricts to $V = V \otimes k \to V \otimes A$, call this function

$\rho$. The map $\rho$ defines on $V$ what is called a *comodule structure* of the Hopf algebra $A$, i.e. a map $\rho : V \to V \otimes A$ such that the following diagram is commutative

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes A \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho\otimes \mathrm{id}} \\ V \otimes A & \xrightarrow{\mathrm{id}\otimes m} & V \otimes A \otimes A \end{array}$$

On the other hand, given a comodule structure $\rho$ on $V$, for every homomorphism of $k$ algebras $r : A \to \mathrm{H}^0(X)$ we can look at the composition $(\mathrm{id} \otimes r) \circ \rho : V \to V \otimes A \to V \otimes \mathrm{H}^0(X)$ and, using the universal property of the tensor product, extend it to an $\mathrm{H}^0(X)$-linear function $V \otimes \mathrm{H}^0(X) \to V \otimes \mathrm{H}^0(X)$. The commutativity of the diagram above ensures that this defines an action of $G$ on the functor $X \mapsto V \otimes \mathrm{H}^0(X)$. We have thus proved the following lemma:

**Lemma 2.31.** *Linear representations of $G = \operatorname{Spec} A$ on $V$ correspond to comodule structures $\rho : V \to V \otimes A$.* □

**Corollary 2.32.** *If $\rho : V \to V \otimes A$ is a comodule, $\rho$ is injective.*

*Proof.* Let $v \in \ker \rho$. Take the identity $\varepsilon_k \in G(k)$ defined by the homomorphism $e : A \to k$. Since $G(k)$ acts on $V \otimes k = V$, $\varepsilon_k(v) = v$. But $\varepsilon_k(v) = (\mathrm{id} \times e) \circ \rho(v) = 0$, hence $v = 0$. □

Let $V$ be a finite dimensional vector space. As we have seen in Example 2.7 and Example 2.8, $V_{sch}$ represents $X \mapsto V \otimes \mathrm{H}^0(X)$, and $\mathrm{GL}(V)$ represents $X \mapsto \mathrm{GL}(V)(X) = \mathrm{GL}_{\mathrm{H}^0(X)}(V \otimes \mathrm{H}^0(X))$ the $\mathrm{H}^0(X)$-linear automorphisms of $V \otimes \mathrm{H}^0(X)$. Hence, thanks to the Yoneda Lemma, representations of $G$ on $V$ are in bijective correspondence with homomorphisms of group-schemes $G \to \mathrm{GL}(V)$.

**Definition 2.33.** Let $G$ be a group-scheme and $V$ a vector space. A representation of $G$ on $V$ is *faithful* if the action of $G(X)$ on $V \otimes \mathrm{H}^0(X)$ is faithful for every scheme $X$ over $k$.

**Lemma 2.34.** *A representation of a group-scheme $G$ on a finite dimensional vector space $V$ is faithful if and only if the associated homomorphism $G \to \mathrm{GL}(V)$ is a closed immersion.*

*Proof.* The representation is faithful if and only if $G(X) \to \mathrm{GL}(V)(X)$ is injective for every scheme $X$, hence the thesis descends immediately from the following lemma. □

**Lemma 2.35.** *A morphism $\varphi : G = \operatorname{Spec} A \to H = \operatorname{Spec} B$ is a closed subgroup if and only if $\varphi(X)$ is injective for every scheme $X$.*

*Proof.* The "only if" part is obvious.

Now consider the splitting $G \to L = \operatorname{Spec} C \to H$, where $L$ is the image of $\varphi$, as defined at the end of 2.1.1. We have that $G \to L$ is a quotient and $L \to H$ a closed immersion. Since $G(X) \to H(X)$ is injective for every $X$, $G(X) \to L(X)$ is injective, too. Now, thanks to Proposition 2.2, we have that $C \subseteq A$ is faithfully flat, hence

$$C \to A \rightrightarrows A \otimes_C A$$

is an equalizer [Vis05, Lemma 2.61]. But, since $G(X) \to L(X)$ is injective, the two projections $L(X) \times_{G(X)} L(X) \to L(X)$ are equal. This implies that the two projections $L \times_G L \to L$ are equal, too, thanks to the Yoneda Lemma. Hence, the arrow $C \to A$ in the equalizer is an isomorphism, and so $G = L$. $\square$

### 2.3.3 Examples

**Example 2.36.** We want to define a representation of the group-scheme $\mu_n = \operatorname{Spec} k[x]/(x^n - 1)$ on $\mathbb{A}^1$. The idea is that, since $\mu_n$ is the group of the roots of unity, it should somehow act by multiplication on $k$. If $X$ is a scheme, $\mu_n(X) = \operatorname{Hom}(k[x]/(x^n-1), \mathrm{H}^0(X))$ may be identified as the set of elements $r \in \mathrm{H}^0(X)$ such that $r^n = 1$. Since $\mathbb{A}^1(X) = \mathrm{H}^0(X)$, we define $\mu_n(X) \times \mathbb{A}^1(X) \to \mathbb{A}^1(X)$ simply by multiplication.

**Example 2.37.** Let $G \times V_{\mathrm{sch}} \to V_{\mathrm{sch}}$ be a representation, and $\rho : V \to V \otimes A$ the associated comodule. We may define the dual representation as usual by $(g \cdot f)(v \otimes r) = f(g^{-1}(v \otimes r))$ for every $g \in G(X)$. Now, let us look at the induced comodule structure. Consider the composition

$$V^\vee \otimes V \xrightarrow{\mathrm{id} \otimes \rho} V^\vee \otimes V \otimes A \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes i} V^\vee \otimes V \otimes A \xrightarrow{\mathrm{ev} \cdot \mathrm{id}} A$$

where $\mathrm{ev} : V^\vee \otimes V \to k$ is the evaluation map $f \otimes v \mapsto f(v)$. The composition $V^\vee \otimes V \to A$ corresponds to a linear map $\rho^\vee : V^\vee \to V^\vee \otimes A$ which is the desired comodule.

**Definition 2.38.** The comultiplication $m : A \to A \otimes A$ defines a comodule structure on $A$: the corresponding linear representation is called the *regular representation* of $G$.

### 2.3.4 Some facts about representations

In this section, in order to make the reading more linear, we have packed some technical results about representations which will be used in the proofs but are not directly involved in the general comprehension.

**Lemma 2.39.** *Let $V$ be a linear representation of an affine group-scheme $G = \operatorname{Spec} A$, with $\rho : V \to V \otimes A$ defining the comodule. Every finite subset $S \subseteq V$ is contained in a finite dimensional subrepresentation of $V$.*

*Proof.* Fix a basis $\{a_i\}$ of $A$. For $v \in S$, write $\rho(v) = \sum_i a_i \otimes w_i^v$ as a finite sum. Since

$$\sum_i a_i \otimes \rho(w_i^v) = (\mathrm{id} \otimes \rho) \circ \rho(v) = (m \otimes \mathrm{id}) \circ \rho(v) =$$

$$= \sum_i m(a_i) \otimes w_i^v = \sum_{ijk} a_j \otimes a_k \otimes r_{ijk} w_i^v$$

and $\{a_i\}$ is a basis, we have $\rho(w_i^v) = \sum_{lk} a_k \otimes r_{lik} w_l^v$. Hence, the subspace of $V$ generated by $S$ and by the vectors $w_i^v$ for $v \in S$ is a subcomodule and hence a subrepresentation. □

**Corollary 2.40.** *Every linear representation of an affine group-scheme is a directed union of finite-dimensional subrepresentations.*

*Proof.* Finite-dimensional subrepresentation form a directed set partially ordered by inclusion with union $V$ thanks to Lemma 2.39. □

**Corollary 2.41.** *An affine group-scheme $G$ is of finite type if and only if it has a faithful representation of finite dimension.*

*Proof.* If $V$ is a faithful representation of finite dimension, it defines an homomorphism $G \to \mathrm{GL}(V)$ which is a closed embedding thanks to Lemma 2.35. Since $\mathrm{GL}(V)$ is of finite type, $G$ is of finite type, too.

On the other hand, if $G = \operatorname{Spec} A$ is of finite type, $A$ is generated as a $k$-algebra by $x_1, \dots, x_n \in A$. Take $V$ a finite subrepresentation of the regular representation containing $x_1, \dots, x_n$. Consider a scheme $X$ and an element $g \in G(X)$ such that $g$ acts as the identity on $V \otimes \mathrm{H}^0(X)$. Since $V$ generates $A$ as a $k$-algebra, and the comodule structure $A \to A \otimes A$ is also an algebra homomorphism, $g$ acts as the identity on $A \otimes \mathrm{H}^0(X)$, too. This implies that

$$(\mathrm{id} \otimes g) \circ \rho = (\mathrm{id} \otimes \varepsilon) \circ \rho : A \to A \otimes A \to A \otimes \mathrm{H}^0(X).$$

Hence, multiplication by $g$ in $G(X)$ is equal to multiplication by $\varepsilon$, and this implies $g = \varepsilon$. □

**Lemma 2.42.** *Every Hopf algebra $A$ is the union of its finitely generated Hopf subalgebras.*

*Proof.* Let $V \subseteq A$ a finite dimensional subcomodule, $\{v_i\}$ a basis of $V$ and $m(v_i) = \sum_j a_{ij} \otimes v_j$. Call $U$ the $k$-subalgebra generated $v_i$, $a_{ij}$, $i(v_i)$ and $i(a_{ij})$, where $i : A \to A$ is the coinverse. It is finitely generated, and coassociativity ensures that it is an Hopf subalgebra as we have done in Lemma 2.39. Since the union of finite subcomodules is the entire $A$ and $V \subseteq U$, we have proved the statement. □

**Corollary 2.43.** *Every affine group-scheme is the limit of its quotients of finite type. This is also true at the level of functors* $\mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Grp}$.

*Proof.* Let $G = \operatorname{Spec} A$ be a group-scheme with quotients of finite type $G_i = \operatorname{Spec} A_i$. Lemma 2.42 implies that $A = \bigcup_i A_i$. Hence, we know that $G = \lim_i G_i$ both as a scheme and as an affine group-scheme. Now take $H : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Grp}$ a functor with a cone $\psi_i : H \to G_i$: we want to show that this defines a unique homomorphisms of group functors $H \to G$.

In order to do this, fix a scheme $T$, we want to define $H(T) \to G(T)$. Take $h \in H(T)$, we have that $\psi_i(h) \in G_i(T)$ defines a cone $T \to G_i$ and hence a unique morphism $T \to G$. This defines a map $H(T) \to G(T)$ that is functorial in $T$ thanks to the functoriality of the cone $H(T) \to G_i(T)$. Hence, we have a well defined natural transformation $H(T) \to G(T)$, we need to check that it is an homomorphism of group functors.

Since the composition $H(T) \to G(T) \to \prod_i G_i(T)$ is $(\psi_i)_i$, which is an homomorphism because $\psi_i$ is an homomorphism for every $i$, it is enough to show that $G(T) \to \prod_i G_i(T)$ is injective. Let $f, g : T \to G$ be different morphisms, they are determined by homomorphisms $A \to \mathrm{H}^0(T)$: since they are different, there exists at least one $i$ such that the restrictions $A_i \to \mathrm{H}^0(T)$ are different, as desired. □

Let $V$ be a representation of $G = \operatorname{Spec} A$, and $G_V = \operatorname{Spec} A_V$ the image of $G \to \mathrm{GL}_V$. If $V \to W$ is an injective $G$-equivariant map of representations, the diagram

$$\begin{tikzcd} & G \arrow[dl, "\rho_V"'] \arrow[dr, "\rho_W"] & \\ G_W \arrow[rr] & & G_V \end{tikzcd}$$

commutes. We need the map $V \to W$ to be injective in order to define $G_W \to G_V$. If $\mathrm{Rep}'_k G$ is the wide subcategory of $\mathrm{Rep}_k G$ with only injective maps, we have defined a functor $\mathrm{Rep}'_k G \to \mathrm{AffGrp}_k$ and a cone $(G, \rho)$.

**Corollary 2.44.** $G = \operatorname{Spec} A$ *is the limit of*

$$\operatorname{Rep}'_k G \to \mathrm{AffGrp}_k \to \operatorname{Hom}(\mathrm{Sch}/k^{\mathrm{op}}, \mathrm{Grp}).$$

*Proof.* Let $\mathcal{Q}_G$ be the category of quotients of finite type of $G$, and $\mathcal{Q}_G \to \mathrm{AffGrp}_k$ the forgetful functor. We have that $\operatorname{Rep}'_k G \to \mathrm{AffGrp}_k$ splits as $\operatorname{Rep}'_k G \to \mathcal{Q}_G \to \mathrm{AffGrp}_k$, and we know that $G$ is the limit of $\mathcal{Q}_G \to \operatorname{Hom}(\mathrm{Sch}/k^{\mathrm{op}}, \mathrm{Grp})$ thanks to Corollary 2.43. We also know that cones of $\mathcal{Q}_G \to \operatorname{Hom}(\mathrm{Sch}/k^{\mathrm{op}}, \mathrm{Grp})$ induce cones of $\operatorname{Rep}'_k G \to \operatorname{Hom}(\mathrm{Sch}/k^{\mathrm{op}}, \mathrm{Grp})$, to conclude we need to show that also the contrary is true.

Let $\psi_V : H \to G_V$ be a cone for $\operatorname{Rep}'_k G \to \operatorname{Hom}(\mathrm{Sch}/k^{\mathrm{op}}, \mathrm{Grp})$. Corollary 2.41 implies that $\operatorname{Rep}'_k G \to \mathcal{Q}_G$ is essentially surjective, we would like to use this fact to define a cone $\varphi_i : H \to G_i$, where $G \to G_i$ is a quotient. If $\lambda : G_V \to G_W$ is an homomorphism of quotients of $G$, we have that $\lambda \circ \psi_V = \psi_W$ because the diagram

$$\begin{tikzcd}
& H \arrow[d, "\psi_{V\oplus W}"] \arrow[ddl, bend right, "\psi_V"'] \arrow[ddr, bend left, "\psi_W"] & \\
& G_{V\oplus W} \arrow[dl] \arrow[dr] & \\
G_V \arrow[rr, "\lambda"] & & G_W
\end{tikzcd}$$

commutes: the lower triangle is composed by maps of quotients of $G$, which are unique, and the other two triangles commute because $\psi$ is a cone. This implies that, if we have an isomorphism $G_V \xrightarrow{\sim} G_i$, $\varphi_i : H \to G_V \to G_i$ is a well defined cone for $\mathcal{Q}_G \to \operatorname{Hom}(\mathrm{Sch}/k^{\mathrm{op}}, \mathrm{Grp})$. □

**Lemma 2.45.** *Every (finite dimensional) linear representation $V$ of an affine group-scheme $G$ embeds into a (finite) direct sum of regular representations.*

*Proof.* After choosing a basis of $V$, we can regard $V \otimes A$ as a direct sum of copies of $A$. Making $G$ act only on $A$, $V \otimes A$ becomes a direct sum of representations.

Let $\rho : V \to V \otimes A$ be the comodule associated to the representation on $V$. Thanks to Corollary 2.32, $\rho$ is an embedding of vector spaces: if we show that $\rho$ is an embedding of representations, too, we are done. To check it, we must show that the following diagram is commutative:

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"] & A \otimes V \arrow[d, "\mathrm{id} \otimes \rho"] \\ A \otimes V \arrow[r, "m \otimes \mathrm{id}"] & A \otimes A \otimes V \end{tikzcd}$$

But this is exactly the condition for $\rho$ to define a comodule. $\square$

Let $V_1, \ldots, V_n$ be representations of a group-scheme $G$. Since tensor products and direct sum of representations of $G$ have a natural induced structure of representation, if $p(x_1, \ldots, p_{x_n})$ is a polynomial in $\mathbb{N}[x_1, \ldots, x_n]$ we may interpret sums as direct sums and products as tensor products in order to define a representation $p(V_1, \ldots, V_n)$.

**Lemma 2.46.** *Let $V$ be a vector space of dimension $d$ and $G = \operatorname{Spec} A$ a closed subgroup of $\mathrm{GL}(V)$. Then every finite dimensional representation $W$ of $G$ is a subrepresentation of a quotient of $p(V, V^\vee)$ for some polynomial $p \in \mathbb{N}[s, t]$.*

*Proof.* Every representation is embedded in a finite sum of copies of $A$ as a representation (Lemma 2.45). Multiplication $G \times \mathrm{GL}(V) \to \mathrm{GL}(V)$ defines a map $\mathcal{O}(\mathrm{GL}(V)) \to \mathcal{O}(\mathrm{GL}(V)) \otimes A$ which is a comodule (associativity of the comodule is ensured by associativity of multiplication in $\mathrm{GL}(V)$). As $A$-comodules, $A$ is a quotient of $\mathcal{O}(\mathrm{GL}(V))$. Moreover, the comodule $\mathcal{O}(\mathrm{GL}(V)) = \mathrm{Sym}(V \otimes V^\vee)_{\det}$ can be thought as a quotient of the comodule

$$\mathrm{Sym}(V \otimes V^\vee) \otimes \mathrm{Sym}(\Lambda^d V^\vee).$$

In fact, $G$ acts on the determinant $\det \in \mathrm{Sym}(V \otimes V^\vee)$ as on $\Lambda^d V = \langle \det \rangle_k$: we only have to identify $\det \otimes \det^{-1} \sim 1$.

To sum up, we know that $W$ is a finite dimensional quotient of a subcomodule of $\mathrm{Sym}(V \otimes V^\vee) \otimes \mathrm{Sym}(\Lambda^d V^\vee)$: this implies our thesis thanks to Lemma 2.39. $\square$

**Lemma 2.47.** *If $G = \operatorname{Spec} A$ is an affine group-scheme of finite type and $H = \operatorname{Spec} B \subseteq G$ is a closed subgroup, there is a finite dimensional representation $V$ of $G$ and a line $L \subseteq V$ such that, for every scheme $X$, $H(X)$ is the subgroup of $G(X)$ sending $L \otimes \mathrm{H}^0(X)$ into itself.*

*Proof.* Call $J \subseteq A$ the kernel of $\varphi : A \to B$. We want to show that $H(X)$ is the subgroup of $G(X)$ sending $J \otimes \mathrm{H}^0(X)$ into itself. If $g \in G(X)$, $g : A \to \mathrm{H}^0(X)$, since $\varphi$ is surjective we have that $g \in H(X)$ if and only if $g$ is 0 on $J$.

Now, let $g \in G(X)$ with $g \cdot (J \otimes \mathrm{H}^0(X)) \subseteq J \otimes \mathrm{H}^0(X)$, this means that $(\mathrm{id} \otimes g) \circ m(J) \subseteq J \otimes \mathrm{H}^0(X)$: we want to show that $g$ is 0 on $J$. Since $g = e \cdot g$, with $e \in H(X)$ the identity, $g : A \to \mathrm{H}^0(X)$ is equal to the composition

$$A \xrightarrow{m} A \otimes A \xrightarrow{\mathrm{id} \otimes g} A \otimes \mathrm{H}^0(X) \xrightarrow{e \otimes \mathrm{id}} \mathrm{H}^0(X) \otimes \mathrm{H}^0(X) \xrightarrow{\Delta} \mathrm{H}^0(X).$$

Using the fact that $g \cdot (J \otimes \mathrm{H}^0(X)) \subseteq J \otimes \mathrm{H}^0(X)$, if we restrict the formula above to $J$ we obtain

$$J \xrightarrow{(\mathrm{id} \otimes g) \circ m} J \otimes \mathrm{H}^0(X) \xrightarrow{e \otimes \mathrm{id}} \mathrm{H}^0(X) \otimes \mathrm{H}^0(X) \xrightarrow{\Delta} \mathrm{H}^0(X)$$

which is 0 because $e \in H(X)$ is 0 on $J$, hence $g \in H(X)$. On the other hand, if $h \in H(X)$, I want to show $h \cdot (J \otimes \mathrm{H}^0(X)) \subseteq J \otimes \mathrm{H}^0(X)$. But $J \otimes \mathrm{H}^0(X)$ is the kernel of $\varphi \otimes \mathrm{id} : A \otimes \mathrm{H}^0(X) \to B \otimes R$, hence it is enough to show that the composition

$$A \xrightarrow{m} A \otimes A \xrightarrow{\mathrm{id} \otimes h} A \otimes \mathrm{H}^0(X) \xrightarrow{\varphi \otimes \mathrm{id}} B \otimes \mathrm{H}^0(X)$$

is 0 on $J$. Since $\varphi$ is an homomorphism of Hopf algebras, this morphism is equal to the following composition

$$A \xrightarrow{\varphi} B \xrightarrow{m} B \otimes B \xrightarrow{\mathrm{id} \otimes h} B \otimes \mathrm{H}^0(X)$$

which is clearly 0 on $J$.

Now, let $W$ be a finite dimensional subrepresentation of $A$ containing a set of generators of $J$ as an ideal: $W$ exists because $A$ is noetherian since it is of finite type over $k$. It is clear that $g \in G(X)$ stabilizes $J \otimes \mathrm{H}^0(X)$ in $A \otimes \mathrm{H}^0(X)$ if and only if it stabilizes $(W \cap J) \otimes \mathrm{H}^0(X)$ in $W \otimes \mathrm{H}^0(X)$. Hence $H(X)$ is the stabilizer of $(J \cap W) \otimes \mathrm{H}^0(X)$ in $W \otimes \mathrm{H}^0(X)$, and finally this implies that it is also the stabilizer of $\Lambda^d(J \cap W) \otimes \mathrm{H}^0(X)$ in $\Lambda^d W \otimes \mathrm{H}^0(X)$, with $d = \dim_k(I \cap W)$. □

## 2.4 Profinite group-schemes

The étale fundamental group was defined by Grothendieck as the projective limit of the automorphism groups of étale coverings. Our approach will be similar, but we are going to deal with the richer structure of group-schemes. In this section, we will develop the theory of profinite group-schemes.

### 2.4.1 Cofiltered diagrams and projective systems

**Definition 2.48.** Let $\mathcal{D}$ be a nonempty partially ordered set. We will say that $\mathcal{D}$ is a *directed set* if, for every pair of objects $a, b \in P$, there exists $c \in \mathcal{D}$ with $a \leq c, b \leq c$.

**Definition 2.49.** If $\mathcal{C}$ is a category, a *direct system* is a diagram in $\mathcal{D} \to \mathcal{C}$ and a *projective system* is a diagram $\mathcal{D}^{\mathrm{op}} \to \mathcal{C}$, where $\mathcal{D}$ is a directed set considered as a category (there exists a unique morphism $a \to b$ if $a \leq b$). A *direct limit* is the colimit of a direct system, a *projective limit* is the limit of a projective system.

Classically, one defines profinite groups as projective limits of finite groups. There is a natural generalization of projective and direct systems giving a more flexible theory.

**Definition 2.50.** Let $\mathcal{D}$ be a category. We will say that $\mathcal{D}$ is *filtered* if

- $\mathcal{D}$ is nonempty.
- For every pair of objects $D_1, D_2$ in $\mathcal{D}$ there exists an object $D$ and morphisms $g_i : D_i \to D$.
- For every pair of morphisms $f_1, f_2 : D \to D'$, there exists an object $D''$ and a morphism $f : D' \to D''$ such that $f \circ f_1 = f \circ f_2$.

**Definition 2.51.** Let $\mathcal{P}$ be a category. We will say that $\mathcal{P}$ is *cofiltered* if $\mathcal{P}^{\mathrm{op}}$ is filtered. Explicitly, if

- $\mathcal{P}$ is nonempty.
- For every pair of objects $P_1, P_2$ in $\mathcal{P}$ there exists an object $P$ and morphisms $g_i : P \to P_i$.
- For every pair of morphisms $f_1, f_2 : P' \to P$, there exists an object $P''$ and a morphism $f : P'' \to P'$ such that $f_1 \circ f = f_2 \circ f$.

Clearly, direct systems are in particular filtered diagrams, and projective systems are cofiltered diagrams.

We now prove the existence of some types of limits and colimits that will be useful later.

**Proposition 2.52.** *The following exist:*

*(i) Small limits of sets.*

*(ii) Small limits of groups.*

*(iii) Small limits of topological groups.*

*(iv) Small and filtered colimits of sets.*

*(v) Small and filtered colimits of commutative rings with identity.*

*(vi) Small and filtered colimits of modules.*

*(vii) Small and filtered colimits of quasi-coherent sheaves.*

*(viii) Small and filtered colimits of Hopf algebras.*

*(ix) Small and cofiltered limits of affine group-schemes.*

*Proof.* (i) Consider a diagram $\mathcal{F} : \mathcal{J} \to \text{Set}$ sending $i$ to the set $S_i$, with $\mathcal{J}$ a small category. Call $S$ the subgroup of $\prod_i S_i$ of elements $(s_i)_{i \in \text{ob}\,\mathcal{J}}$ such that $\mathcal{F}(f)(s_j) = s_i$ for all maps $f : j \to i$ in $\mathcal{P}$. We claim that $S$ is $\lim_i S_i$.

The canonical projections $\pi_i : S \to S_i$ make $(S, \pi)$ a cone: we want to show that it is universal. Let $(T, \varphi)$ be another cone, we may regard the family of maps $\varphi$ as a map $\varphi : T \to \prod_i S_i$: the fact that $(T, \varphi)$ is a cone implies that the image of $\varphi$ is contained in $S$, hence defining a morphism of cones $(T, \varphi) \to (S, \pi)$. This is unique: if $h : T \to S$ defines a morphism of cones, the composition of $h$ with the injection $S \to \prod_{i \in \text{ob}\,\mathcal{P}} S_i$ must be equal to $\varphi$.

(ii) Consider a diagram $\mathcal{F} : \mathcal{J} \to \text{Grp}$ sending $i$ to the group $G_i$, with $\mathcal{J}$ a small category. Call $G$ the limit as sets $\lim_i G_i$, $G \subseteq \prod_i G_i$ inherits the structure of group and the projections $G \to G_i$ are clearly homomorphism defining a cone for $\mathcal{F}$. If $(H, \pi)$ is another cone, since $G = \lim_i G_i$ as sets there is a unique map of sets $H \to G$ which is easily verified to be an homomorphism. Hence, $G$ is the limit $\lim_i G_i$ as groups.

(iii) Consider a diagram $\mathcal{F} : \mathcal{J} \to \text{TopGrp}$ sending $i$ to the topological group $G_i$, with $\mathcal{J}$ a small category. Now call $G$ the limit of $\mathcal{F}$ composed with the forgetful functor $\text{TopGrp} \to \text{Grp}$, it is the limit of the groups $G_i$ without topology. We want to put on $G$ a topology making it the limit of $\mathcal{F}$.

Hence, consider the natural projections $\pi_i : G \to G_i$, and put on $G$ the coarsest topology making $\pi_i$ continuous for every $i \in \text{ob}\,\mathcal{J}$. We check that multiplication on $G$ is continuous, the inverse is analogous.

To check that $m : G \times G \to G$ is continuous, it is enough to check that the composition $G \times G \to G_i$ is continuous for every $i$. But this map splits as $G \times G \to G_i \times G_i \to G_i$, and both this maps are continuous by definition.

Now, $(G, \pi)$ is a cone of $\mathcal{F}$. Let $(H, \varphi)$ be another cone, we want to show that there exists a unique morphism of cones $(H, \varphi) \to (G, \pi)$. Clearly, there exists a unique possible homomorphism of groups $H \to G$, we need to check that it is continuous. But this is obvious, because all compositions $H \to G \to G_i$ are continuous.

(iv) Let $\mathcal{F} : \mathcal{D} \to \text{Set}$ be a direct system sending an object $i$ to the set $S_i$, and consider the disjoint union $\bigsqcup_i S_i$. On $\bigsqcup_i S_i$ define the following relation: $a \sim b$ if there exist maps $f, f'$ in $\mathcal{D}$ such that $\mathcal{F}(f)(a) = \mathcal{F}(f')(b)$. The relation is clearly symmetric and reflexive, and it is also transitive because $\mathcal{D}$ is filtered. Call $S$ the quotient $\bigsqcup_i S_i / \sim$, we want to show that $S$ is the limit of $\mathcal{F}$.

The compositions $\psi_i : S_i \to \bigsqcup_i S_i \to S$ make $(S, \psi)$ a co-cone. If $(T, \varphi)$ is another co-cone, there is a unique function $\bigsqcup_i S_i \to T$ making the following diagram commutative:

$$\begin{tikzcd} S_i \arrow[rr, "\varphi_i"] \arrow[dr] & & T \\ & \bigsqcup_i S_i \arrow[ur] & \end{tikzcd}$$

and clearly, since $(T, \varphi)$ is a co-cone, the map $\bigsqcup_i S_i \to T$ passes to the quotient as $S \to T$.

(v) Let $\mathcal{F} : \mathcal{D} \to \text{CRing}$ be a direct system sending an object $i$ to the ring $A_i$, call $A$ the colimit $\operatorname{colim}_i A_i$ as sets. We claim that $A$ inherits the structure of a commutative ring with identity.

The fact that $\mathcal{D}$ is filtered ensures that the operations on $A$ are well defined. Take $[a], [b]$ two equivalence classes in $A$: since $\mathcal{D}$ is filtered we may suppose that $a, b$ are contained in the same ring $A_i$ and hence we may define $[a] + [b] = [a + b]$ and $[a] \cdot [b] = [a \cdot b]$. If $a', b' \in A_j$ and $a' = \mathcal{F}(f)(a)$, $b' = \mathcal{F}(g)(b)$ with $f, g : i \to j$, since $\mathcal{D}$ is filtered we may suppose $f = g$ up to composing them with another homomorphism, and hence $a' + b' = \mathcal{F}(f)(a + b)$, $a' \cdot b' = \mathcal{F}(f)(a \cdot b)$. Commutativity of $A$ is obvious, and the class of the identity of some ring $A_i$ is an identity for $(A, \cdot)$.

Now, we want to show that $A$ is the limit of $\mathcal{F}$. The compositions $\psi_i : A_i \to \bigsqcup_i A_i \to A$ are clearly homomorphisms making $(A, \psi)$ a co-cone. If $(B, \varphi)$ is another co-cone, since $A$ is the limit as sets there is a unique morphism of co-cones of sets $(A, \psi) \to (B, \psi)$, and the function $A \to B$ is an homomorphism.

(vi) Let $\mathcal{F} : \mathcal{D} \to \mathrm{Mod}_R$ be a direct system sending $i$ to the $R$-module $M_i$. Then the colimit as sets $M = \operatorname{colim}_i M_i$ has a natural structure of $R$-module making it the colimit of $\mathcal{F}$. The proof is completely analogous to the one about commutative rings.

(vii) Let $\mathcal{F} : \mathcal{D} \to \mathrm{QCoh}(X)$ be a direct system sending $i$ to the quasi-coherent sheaf $S_i$ over $X$. Call $S$ the presheaf $U \mapsto \operatorname{colim}_i S_i(U)$ and $S^{\mathrm{sh}}$ the sheafification of $S$. It is easy to check that $S^{\mathrm{sh}}$ is the colimit at the level of sheaves: if $(Q, \alpha)$ is a co-cone for $\mathcal{F}$, we have a unique map of presheaves $S \to Q$ inducing a unique map of sheaves $S^{\mathrm{sh}} \to Q$. We need to check that $S^{\mathrm{sh}}$ is quasi-coherent.

*Step 1*: if $U \subseteq X$ is a quasi-compact open subset, the natural map $\operatorname{colim}_i S_i(U) \to S^{\mathrm{sh}}(U)$ is an isomorphism.

Take a section $s \in S^{\mathrm{sh}}(U)$, by definition of sheafification $s$ is defined by a covering $U = \bigcup_j U_j$ and by sections $s_j \in \operatorname{colim}_i S_i(U_j)$ such that $s_j|_{U_{jk}} = s_k|_{U_{jk}}$. Since $U$ is quasi-compact, we may suppose that the covering is finite. Now, since there is a finite number of open sets $U_j$ and a finite number of intersections $U_{jk}$, we may find an object $i_0$ in $\mathcal{D}$ and a section $s_0 \in S_i(U)$ such that the image of $s_0|_{U_j}$ in $\operatorname{colim}_i S_i(U_j)$ is $s_j$, hence $\operatorname{colim}_i S_i(U) \to S^{\mathrm{sh}}(U)$ is surjective. To verify that it is injective, let us suppose that the image of $s_0$ in $\operatorname{colim}_i S_i(U)$ is 0, this means that there exists an object $i_1$ in $\mathcal{D}$ and a morphism $i_0 \to i_1$ such that the image of $s_0$ in $S_{i_1}(U)$ is 0. This implies that for every $s_j$ is 0 for every $j$, too, and hence $s = 0$.

*Step 2*: if $U \subseteq X$ is an affine open subset, $S^{\mathrm{sh}}|_U \simeq \widetilde{S^{\mathrm{sh}}(U)}$.

If $M$ is an $R$-module and $f \in R$, $M_f = M \otimes_R R_f$, hence localization is a colimit and commutes with colimits. Since affine schemes are quasi-compact, for every $f \in \mathrm{H}^0(U)$ we have that

$$S^{\mathrm{sh}}(U_f) = \operatorname*{colim}_i (S_i(U_f)) = \operatorname*{colim}_i (S_i(U)_f) =$$

$$= \operatorname*{colim}_i (S_i(U))_f = S^{\mathrm{sh}}(U)_f$$

and hence $S^{\mathrm{sh}}|_U \simeq \widetilde{S^{\mathrm{sh}}(U)}$.

(viii) Let $\mathcal{F} : \mathcal{D} \to \mathrm{Hopf}_k$ be a direct system sending an object $i$ to the Hopf algebra $A_i$. Define $A$ as above as a limit of rings, we need to check that $A$ inherits the structure of Hopf algebra. Since $A = \mathrm{colim}_i A_i$ as rings, to define an homomorphism of rings $A \to A \otimes A$ is enough to give a co-cone $(A \otimes A, \varphi)$. Hence, define $\varphi_i : A_i \to A \otimes A$ as the composition

$$A_i \xrightarrow{\rho_i} A_i \otimes A_i \to A \otimes A$$

where $\rho_i$ is the comultiplication of $A_i$.

This defines a co-cone because if $f$ is a morphism in $\mathcal{D}$, $\mathcal{F}(f)$ is a morphism of Hopf algebras:

$$\begin{tikzcd}
A_i \arrow[r, "\rho_i"] \arrow[d, "\mathcal{F}(f)"'] & A_i \otimes A_i \arrow[d] \arrow[dr] & \\
A_j \arrow[r, "\rho_j"] & A_j \otimes A_j \arrow[r] & A \otimes A
\end{tikzcd}$$

and hence we have an homomorphism $\rho : A \to A \otimes A$. Coinverse and coidentity of $\rho$ are induced similarly by the respective homomorphisms on $A_i$ for every $i$, and they respect the necessary restrictions for the same reason. Clearly, the co-cone of rings on $A$ becomes a co-cone of Hopf algebras, too. If $(B, \psi)$ is another co-cone of Hopf algebras, since $A = \mathrm{colim}_i A_i$ as rings this defines a unique homomorphism of rings $A \to B$, which is a morphism of Hopf algebras, too, because $(B, \psi)$ is a co-cone of Hopf algebras.

(ix) This is a direct consequence of point (viii).

□

**Proposition 2.53.** *Let $\mathcal{P}$ be a small, cofiltered category, $\mathcal{J}$ a finite category and $\mathcal{F} : \mathcal{P} \times \mathcal{J} \to \mathrm{Set}$ a functor $(p, j) \mapsto S_{p,j}$. Then, the natural map $\lambda : \mathrm{colim}_j \lim_p S_{p,j} \to \lim_p \mathrm{colim}_j S_{p,j}$ is a bijection.*

*Proof.* [Bor94, Theorem 2.13.4]. □

### 2.4.2 Profinite groups

We have defined a profinite group as a projective limit of finite groups. Clearly, a projective system is in particular a small, cofiltered diagram, hence profinite group are small, cofiltered limits of finite groups: we will now prove the converse.

**Proposition 2.54.** *Small, cofiltered limits of finite groups are profinite.*

*Proof.* Let $G$ be given as the limit of a small, cofiltered diagram $\mathcal{C} \to$ TopGrp, $i \mapsto G_i$ with $G_i$ finite. For every $i \in \mathcal{C}$, call $G_i'$ the image of $G$ in $G_i$. Since $G \to G_i'$ is surjective, every pair of homomorphisms $G_i \to G_j$ restrict to a unique homomorphism $G_i' \to G_j'$, inducing a natural preorder on the set $\{G_i'\}_i$ ($G_j' \geq G_i'$ if exists $G_j' \to G_i'$). We have thus an equivalence relation on $\{G_i'\}_i$ ($G_i' \sim G_j'$ if $G_j' \geq G_i'$ and $G_i' \geq G_j'$) whose equivalence classes are made of isomorphic finite groups. The fact that $\mathcal{C}$ is cofiltered implies that the partial order on $\{G_i\}_i/\sim$ is projective, and the construction of Proposition 2.52.(iii) shows that the projective limit of $\{G_i\}_i/\sim$ is exactly $G$. $\square$

Now, we want to prove that every profinite group has a natural structure of group-scheme, with a construction extending the one of discrete groups given in Example 2.5.

**Lemma 2.55.** *Profinite groups are compact and Hausdorff.*

*Proof.* We will show in general that small limits of compact, Hausdorff groups are compact and Hausdorff.

Let $G$ be a topological group given as the limit of $\mathcal{F} : \mathcal{P} \to$ TopGrp, $i \mapsto G_i$ with $G_i$ compact and Hausdorff. Thanks to the construction given in Proposition 2.52, $G$ is a subgroup of $\prod_i G_i$. Thanks to Tychonoff's theorem, $\prod_i G_i$ is compact, and since $G_i$ is Hausdorff $\prod_i G_i$ is Hausdorff, too. Therefore, it is enough to show that $G \subseteq \prod_i G_i$ is closed.

Let $(g_i)_i \in \prod_i G_i$ be an element not contained in $G$: there are objects $i, j$ and a morphism $f : j \to j'$ in $\mathcal{P}$ such that $\mathcal{F}(f)(g_j) \neq g_{j'}$. But then $U = \{(h_i)_i \in \prod_i G_i | h_j = g_j, h_{j'} = g_{j'}\} \subseteq \prod_i G_i$ is an open subset containing $(g_i)_i$ such that $U \cap G = \emptyset$. $\square$

Let $G$ be a profinite group. Put on $k$ the discrete topology, and consider the set $k^G$ of continuous functions $G \to k$, the structure of field of $k$ induces a natural structure of commutative ring with identity on $k^G$.

**Lemma 2.56.** *Continuous functions $G \to k$ separate points. More precisely, if $p, q \in G$, $p \neq q$, there exists a continuous $f : G \to k$ with $f(G) \subseteq \{0, 1\}$ such that $f(p) = 0$ and $f(q) = 1$.*

*Proof.* Let $G$ be $\lim_i G_i$ with $G_i$ finite group: since $p \neq q$, there exists $i$ with $\pi_i(p) \neq \pi_i(q)$, where $\pi_i : G \to G_i$ is the natural projection. Let us define a function $f'$ on $G$ by $f(\pi_i(q)) = 1$ and $0$ otherwise: we have that $f = f \circ \pi_i : G \to k$ is continuous, $f(p) = 0$ and $f(q) = 1$. □

Now, take a point $p \in G$, the subset

$$\mathfrak{m}_p = \{f \in k^G | f(p) = 0\}$$

is clearly an ideal. Since $f \mapsto f(p)$ defines an isomorphism $k^G/\mathfrak{m}_p \simeq k$, $\mathfrak{m}_p$ is maximal.

**Proposition 2.57.** *The map $p \mapsto \mathfrak{m}_p$ from $G$ to* $\operatorname{Spec} k^G$ *is an homeomorphism.*

*Proof.* Since for every $p, q \in G$, $p \neq q$, there exists $f$ with $f(p) = 0$ and $f(q) \neq 0$ thanks to Lemma 2.56, $G \to \operatorname{Spec} k^G$ is injective. Let us show that it is also surjective. Take a prime ideal $\mathfrak{p} \subseteq k^G$. For every $f \in \mathfrak{p}$, there zero set $V(f)$ is nonempty: if it were empty, $1/f$ would be a function of $k^G$, and hence $\mathfrak{p} = k^G$, absurd. Since $G$ is compact,

$$V(\mathfrak{p}) = \bigcap_{f \in \mathfrak{p}} V(f)$$

is equal to $V(f_1) \cap \cdots \cap V(f_n) = V(f_1 \cdots f_n)$ for some $f_1, \ldots, f_n \in k^G$. Up to a replacing $f_1$ with $f_1 \cdots f_n$, we may suppose $n = 1$. Let $g$ be defined by $g|_{V(f_1)} = 1$ and $g = 1/f_1$ otherwise: up to replacing $f_1$ with $gf_1$, we may suppose $f_1 = 1$ outside of $V(\mathfrak{p})$. Now, if $f|_{\mathfrak{p}} = 0$, we have $f = f_1 \cdot f \in \mathfrak{p}$, and hence $\mathfrak{p} = \{f \in k^G | f|_{V(\mathfrak{p})} = 0\}$. Note that we have only used the fact that the ideal $\mathfrak{p}$ is nontrivial, we will use the fact that it is prime to show that $V(\mathfrak{p})$ has only one point.

We already know that $V(\mathfrak{p}) = V(f_1)$ is nonempty. Let $p, q \in G$ be different points, we have already proved in Lemma 2.56 that there exists $f \in k^G$ with $f(p) = 0$, $f(q) = 1$ and $f(G) \subseteq \{0, 1\}$. Now, since $f(G) \subseteq \{0, 1\}$, $f \cdot (1 - f) = f - f^2 = 0 \in \mathfrak{p}$, and hence $f \in \mathfrak{p}$ or $1 - f \in \mathfrak{p}$ because $\mathfrak{p}$ is prime. This implies that at least one between $p$ and $q$ is not contained in $V(\mathfrak{p})$.

We want now to prove that $G \to \operatorname{Spec} k^G$ is continuous. Let $I \subseteq k^G$ be an ideal, it is enough to show that the closed subset $\operatorname{Spec} k^G/I \subseteq \operatorname{Spec} k^G$ corresponds to a closed subset of $G$. As noted above there exists $f_I \in k^G$

such that $V(I) = V(f_I)$: this means that $V(I) = f_I^{-1}(0) \subseteq G$ is closed, and the bijection $G \to \operatorname{Spec} k^G$ identifies $V(I)$ and $\operatorname{Spec} k^G/I$.

Now, $\operatorname{Spec} k^G$ is Hausdorff: if $p \neq q \in G$, $f(p) = 0$, $f(q) = 1$ and $f(G) \in \{0,1\}$, we have that $\operatorname{Spec} k^G = \operatorname{Spec} k^G_f \sqcup \operatorname{Spec} k^G_{1-f}$, $\mathfrak{m}_p \in \operatorname{Spec} k^G_{1-f}$ and $\mathfrak{m}_q \in \operatorname{Spec} k^G_f$. Hence, since $G$ is compact and $\operatorname{Spec} k^G$ is Hausdorff, the map $G \to \operatorname{Spec} k^G$ is also closed, and hence it is an homeomorphism. $\square$

**Lemma 2.58.** *There exists a natural structure of group-scheme on* $\operatorname{Spec} k^G$ *compatible with the structure of group of G, i.e. the following diagram of sets commutes:*

$$\begin{array}{ccc} \operatorname{Spec} k^G \times_k \operatorname{Spec} k^G & \longrightarrow & \operatorname{Spec} k^G \\ \downarrow & & \downarrow \\ G \times G & \longrightarrow & G \end{array}$$

*Proof.* Fix a function $f \in k^G$, since $G$ is compact and $k$ has the discrete topology there exists a finite quotient $G \to G_i$ such that $f$ descends to $f_i : G_i \to k$. Since $k^{G_i} \otimes k^{G_i} = k^{G_i \times G_i}$, we may define $\rho_i(f_i) \in k^{G_i} \otimes k^{G_i}$ as

$$\rho_i(f_i)(g,h) = f_i(gh)$$

and then define $\rho(f) \in k^G \otimes k^G$ as the pullback of $\rho_i(f_i)$. It is easy to see that $\rho(f)$ doesn't depend on what quotient $G \to G_i$ we use, and hence it defines a map $\rho : k^G \to k^G \otimes k^G$ which is an homomorphism because $\rho_i$ is an homomorphism for every quotient. One may also define $\varepsilon : k^G \to k$ as $\varepsilon(f) = f(e)$ and $i : k^G \to k^G$ as $i(f)(g) = f(g^{-1})$. These constructions define a structure of Hopf algebra on $k^G$, which is clearly compatible with the structure of group of $G$. $\square$

Now, let $G \to H$ be a continuous homomorphism of profinite groups. The pullback $k^H \to k^G$ defines an homomorphism of Hopf algebras, extending the association $G \mapsto \operatorname{Spec} k^G$ to a functor from the category PFGrp of profinite groups to $\mathrm{AffGrp}_k$.

**Proposition 2.59.** *The functor* $G \mapsto \operatorname{Spec} k^G$ *is fully faithful.*

*Proof.* Let $\varphi_1, \varphi_2 : G \to H$ two different continuous homomorphisms of profinite groups. There exists $g \in G$ such that $\varphi_1(g) \neq \varphi_2(g)$. Take $f \in k^H$ such that $f(\varphi_1)(g) \neq f(\varphi_2(g))$, then $\varphi_1^* f \neq \varphi_2^* f$ and hence the functor is faithful. Let us show that is full.

Let $\varphi : \operatorname{Spec} k^G \to \operatorname{Spec} k^H$ be an homomorphism of affine group-schemes, since $G \simeq \operatorname{Spec} k^G$ as topological spaces, $\varphi$ define a continuous map $G \to H$. Moreover, the fact that $\varphi$ is an homomorphism implies that $G \to H$ is an homomorphism. In fact, the diagram

$$\begin{tikzcd}
& \operatorname{Spec} k^H \times_k \operatorname{Spec} k^H \arrow[rr] \arrow[dd, dotted] & & \operatorname{Spec} k^H \arrow[dd] \\
\operatorname{Spec} k^G \times_k \operatorname{Spec} k^G \arrow[ur] \arrow[rr] \arrow[dd] & & \operatorname{Spec} k^G \arrow[ur] \arrow[dd] & \\
& H \times H \arrow[rr, dotted] & & H \\
G \times G \arrow[ur, dotted] \arrow[rr] & & G \arrow[ur] &
\end{tikzcd}$$

commutes because $\operatorname{Spec} k^G \times_k \operatorname{Spec} k^G \to G \times G$ is surjective and thanks to Lemma 2.58. Finally, one may check that $G \to H$ induces $\varphi : k^G \to k^H$. □

Profinite groups satisfy a particular property: if $\{G_i\}_i$ is a projective system of finite groups and $H$ is a finite group, every morphism $\lim_i G_i \to H$ splits as $\lim_i G_i \to G_j \to H$ for some $j$. This means that the pro-category of finite groups (the category of projective systems of groups) is equivalent to the category of profinite groups. We will prove this fact more generally for limits of affine group-schemes of finite type.

**Proposition 2.60.** *Let $\mathcal{F} : \mathcal{P} \to \mathrm{AffGrp}_k$ be a cofiltered diagram of affine group-schemes sending $i$ to $G_i = \operatorname{Spec} A_i$, and let $\operatorname{Spec} A = G = \lim_i G_i$ be the limit. If the group-schemes $G_i$ are of finite type, every homomorphism $G \to H = \operatorname{Spec} B$ to a group-scheme of finite type $H$ splits as $G \to G_i \to H$ for some $i$.*

*Proof.* Let $f : B \to A$ be the morphism of Hopf algebras defined by $G \to H$. Let $b_1, \dots, b_n$ generate $B$ as a $k$-algebra: since $\mathcal{P}$ is cofiltered, there exists $A_1$ in the direct system such that $f(b_j) \in \operatorname{im}(\psi_1)$ for every $j = 1, \dots, n$, where $\psi_1 : A_1 \to A$ is the morphism induced by the fact that $A = \operatorname{colim}_i A_i$.

The identification $x_i \mapsto b_i$ for $i = 1, \dots, n$ defines an homomorphism $b : k[x_i] \to B$ with kernel an ideal $J$, which is finitely generated by $p_1, \dots, p_m$ because $k[x_i]$ is noetherian. Hence, we have a well defined morphism $\gamma : k[x_i] \to A_1$ such that the following diagram is commutative:

$$\begin{tikzcd}
k[x_i] \arrow[r, "b"] \arrow[d, "\gamma"] & B \arrow[d, "f"] \\
A_1 \arrow[r, "\psi_1"] & A
\end{tikzcd}$$

We would like to find an Hopf algebra $A_2$ in the direct system and an homomorphisms of rings $f_2 : B \to A_2$ making the following diagram commutative:

$$\begin{tikzcd}
k[x_i] \arrow[rr, "b"] \arrow[d, "\gamma"'] & & B \arrow[dl, "f_2"'] \arrow[d, "f"] \\
A_1 \arrow[r, "\psi_{2,1}"'] & A_2 \arrow[r, "\psi_2"'] & A
\end{tikzcd}$$

In order to do this, it is enough to find $A_2$ and $\psi_{2,1} : A_1 \to A_2$ such that $\psi_{2,1}(\gamma(p_j)) = 0$ for every $j = 1, \ldots, m$. But $A_2, \psi_{2,1}$ exist because

$$\psi_1(\gamma(p_j)) = f(b(p_j)) = f(p_j(b_i)) = 0$$

and the system is directed. Finally, we need to find an Hopf algebra $A_3$ in the direct system with a morphism $\psi_{3,2} : A_2 \to A_3$ such that $f_3 = \psi_{3,2} \circ f_2 : B \to A_3$ is not only an homomorphism of rings, but also a morphism of Hopf algebras.

$$\begin{tikzcd}
B \arrow[d, "f_2"'] \arrow[dr, "f_3"] \arrow[drr, "f"] & & \\
A_2 \arrow[r, "\psi_{3,2}"'] & A_3 \arrow[r, "\psi_3"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r, "f_2"] \arrow[d] & A_3 \arrow[r, "\psi_3"] \arrow[d] & A \arrow[d] \\
B \otimes B \arrow[r] & A_3 \otimes A_3 \arrow[r] & A \otimes A
\end{tikzcd}$$

Let $m_2, m_B$ be the comultiplications respectively of $A_2$ and $B$: our problem is that, in general, $f_2 \otimes f_2(m_B(b)) \neq m_2(f_2(b))$ for $b \in B$. We need to find $A_3$ and $f_3$ such that this is true. Since $A \otimes A = \operatorname{colim}_i A_i \otimes A_i$, we can find $A_3$ such that $f_3 \otimes f_3(m_B(b)) = m_3(f_3(b))$ for a finite number of elements of $B$, in particular for a finite system of generators. But $f_3$ is an homomorphism of rings: hence the fact that $f_3 \otimes f_3 \circ m_B = m_3 \circ f_3$ holds on a system of generators implies that it holds on every $b \in B$. $\square$

# Chapter 3

# Actions of group-schemes

Grothendieck defined in [SGA1] the étale fundamental group of a scheme $X$ with a geometric base point as the group of automorphisms of the fibre functor on the category of étale coverings of the base scheme (for a detailed exposition, see [Mur67]). What we are going to do is very similar, and the idea behind the definition of Nori's fundamental group-scheme is almost the same. The main difference is that, instead of étale coverings, we are going to use principal bundles, which are called torsors in the algebraic context: this gives use "coverings" with fibers that, instead of being simply finite sets, have a richer structure of group-schemes.

## 3.1 Descent Theory

In order to study torsors, we need some facts of descent theory that will let us work locally, where locally means on a fpqc covering. There are a lot of constructions and proofs that become simpler when done on a covering, and descent theory let us "carry" them down to the base. Here we will only give definitions and results without proofs, for further reading see [Vis05].

### 3.1.1 Fpqc morphisms

**Definition 3.1.** A morphism of schemes $f : X \to Y$ is *fpqc*, *fidèlement plat et quasi-compact*, if it is faithfully flat and every $x \in X$ has an open neighbourhood $U$ such that $f(U)$ is open and $f|_U : U \to f(U)$ is quasi-compact.

**Definition 3.2.** An fpqc covering of a scheme $X$ is a collection of morphisms $\{\sigma_i : U_i \to X\}_{i \in I}$ such that $\sigma : \coprod_i U_i \to X$ is fpqc.

*Remark* 3.3. Fpqc morphisms are not simply, as the name may suggest, faithfully flat and quasi-compact morphisms: fpqc is a slightly weaker condition. If a morphism $f$ is faithfully flat and quasi-compact then it is fpqc but, following our definition, the converse is not true. This is because, in order to make fpqc topology behave well, we want the condition of quasi-compactness to be local. For example, we want the collection $\{\operatorname{Spec} k_i \to \operatorname{Spec} k\}_{i\in I}$, where $k$ is a field and $k_i$ is a copy of $k$, to be a fpqc covering even if in general $\coprod_i \operatorname{Spec} k_i \to \operatorname{Spec} k$ is not quasi-compact.

**Lemma 3.4.** *Let $f : X \to Y$ be an fpqc morphism. Then, $U \subseteq Y$ is open if and only if $f^{-1}(U)$ is open.*

*Proof.* The condition is clearly local in the domain, hence we may suppose that $f$ is quasi-compact. If $f$ is quasi-compact, we may apply [EGAIV-2, Corollaire 2.3.12]. □

**Lemma 3.5.** *A faithfully flat morphism $f : X \to Y$ is fpqc if and only if every quasi-compact open subset of $Y$ is the image of some quasi-compact open subset of $X$.*

*Proof.* Let $f$ be fpqc and $V \subseteq Y$ be open and quasi-compact. Let $x_1 \in X$ be a point such that $f(x_1) \in V$, there exists $U_1 \subseteq X$, $x_1 \in U_1$ such that $f(U_1)$ is open and $f|_{U_1} : U_1 \to f(U_1)$ is quasi-compact. Now, if $V$ is not contained in $f(U_1)$, take $x_2 \in X$ such that $f(x_2) \in V \setminus f(U_1)$, and repeat the construction. Since $V$ is quasi-compact, there exists a finite $n$ such that $V \subseteq f(U) = f(U_1 \cup \cdots \cup U_n)$, and $f|_U : U \to f(U)$ is quasi-compact. Hence, $V$ is the image of $f|_U^{-1}(V)$, which is quasi-compact.

On the other hand, take $x \in X$ and an affine open neighbourhood $V \subseteq Y$ of $f(x)$. There exists a quasi-compact open $U \subseteq X$ with $f(U) = V$, and call $U' \subseteq f^{-1}(V)$ an affine open neighbourhood of $x$. Then $U'' = U \cup U'$ is quasi-compact, $x \in U''$, $f(U'') = V$ and $f|_{U''} : U'' \to V$ is quasi-compact because $V$ is affine and $U''$ is quasi-compact. □

**Proposition 3.6.** *(i) An isomorphism $U \xrightarrow{\sim} X$ is an fpqc covering.*

*(ii) Let $\{\sigma_i : U_i \to X\}$ and $\{\tau_{i,j} : V_{i,j} \to U_i\}$ be fpqc coverings. Then, $\{\sigma_i \circ \tau_{i,j} : V_{i,j} \to X\}$ is an fpqc covering.*

*(iii) Let $\{U_i \to X\}$, be a fpqc covering, and $Y \to X$ a morphism. Then, $\{U_i \times_X Y \to Y\}$ is an fpqc covering.*

*Proof.* (i) Obvious.

(ii) Clearly, $\coprod_{ij} V_{i,j} \to X$ is faithfully flat. If $Y \subseteq X$ is a quasi-compact open set, then there exists a quasi-compact open set $\bigsqcup_i Y_i \subseteq \bigsqcup_i U_i$, with $Y_i \subseteq U_i$, such that $Y$ is the image of $\bigsqcup_i Y_i$. But $\bigsqcup_i Y_i$ is quasi-compact, hence $Y_i$ is empty except for a finite number of indices $i$. For every nonempty $Y_i$, consider a quasi-compact open subset $\bigsqcup_j Y_{i,j} \subseteq \bigsqcup_j V_{i,j}$ with image $Y_i$. Since there is only a finite number of indices such that $Y_i$ is nonempty, $\bigsqcup_{i,j} Y_{i,j}$ is quasi-compact, too, and its image in $X$ is exactly $Y$.

(iii) Clearly, $\coprod_i Y \times_X U_i \to Y$ is faithfully flat. Now take a point $s \in Y \times_X U_i$, with $p(s)$ its image in $U_i$. Since $\sigma : \coprod_i U_i \to X$ is fpqc, there exists $U \subseteq \coprod_i U_i$ open neighbourhood of $p(s)$ with $\sigma(U)$ open and $\sigma|_U : U \to \sigma(U)$ quasi-compact, and up to a replace I may also suppose $U$ quasi-compact. Then $U$ intersects $U_i$ only for a finite number of indices, and hence $Y \times_X U$ is an open subset of $\coprod_i Y \times U_i$ containing $s$. Moreover, the image of $Y \times_X U$ in $Y$ is $f^{-1}(\sigma(U))$, which is open, and $Y \times_X U \to f^{-1}(\sigma(U))$ is quasi-compact because it is the pullback of $\sigma|_U : U \to \sigma(U)$.

□

The properties we have proved in Proposition 3.6 are the ones defining a *Grothendieck topology*.

**Proposition 3.7.** *Let $Y \to X$ be a morphism of schemes over a base scheme $S$. Let $S' \to S$ be a faithfully flat and quasi compact morphism, and*

$$Y' = Y \times_S S' \to X \times_S S' = X'$$

*the base change. Suppose that $Y' \to X'$ has one of the following properties:*

*(i) is surjective,*

*(ii) is quasi-compact,*

*(iii) is locally of finite presentation,*

*(iv) is an isomorphism,*

*(v) is of finite type,*

*(vi) is affine,*

*(vii) is finite,*

*(viii) is flat,*

*(ix) is unramified,*

*(x) is étale.*

*Then $Y \to X$ has the same property.*

*Proof.* Statements from (i) to (viii) are proved in 2.6.1, 2.6.4 and 2.7.1 in [EGAIV-2].

We want now to prove statement (ix). Let $f : Y' \to X'$ be unramified, we want to show that $f : Y \to X$ is unramified, too. Thanks to statement (iii), $f$ is locally of finite presentation. Let $K$ be a field and consider a morphism $\operatorname{Spec} K \to X$, thanks to Lemma 2.11 it is enough to show that $Y \times_X \operatorname{Spec} K$ has the discrete topology and is reduced. Since $X' \to X$ is surjective, there exists a field extension $K'/K$ and a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & X \end{array}$$

We know that $Y' \times_{X'} \operatorname{Spec} K'$ has the discrete topology and is reduced. Moreover, we have a faithfully flat and quasi-compact morphism $Y' \times_{X'} \operatorname{Spec} K' \to Y \times_X \operatorname{Spec} K'$ (it is a base change of $X' \to X$), hence $Y \times_X \operatorname{Spec} K'$ has the discrete topology and is reduced thanks to [EGAIV-2, Corollaire 2.3.12] and to the fact that $\mathcal{O}_{Y \times_X \operatorname{Spec} K'} \subseteq \mathcal{O}_{Y' \times_{X'} \operatorname{Spec} K'}$. For the same reason, $Y \times_X \operatorname{Spec} K$ is reduced and has the discrete topology using the faithfully flat and quasi-compact morphism $Y \times_X \operatorname{Spec} K' \to Y \times_X \operatorname{Spec} K$.

Finally, statement (x) is a direct consequence of statements (viii) and (ix). □

### 3.1.2 Descent data

**Definition 3.8.** Let $\mathcal{U} = \{\sigma_i : U_i \to X\}_i$ be a fpqc covering. We will say that a collection of morphism $f_i : U_i \to Y$ is a morphism $\mathcal{U} \to Y$ if, for every $i, j$, the following diagram is commutative:

$$\begin{tikzcd} U_i \times_X U_j \arrow[r] \arrow[d] & U_j \arrow[d, "f_j"] \\ U_i \arrow[r, "f_i"] & Y \end{tikzcd}.$$

We will call $\mathrm{Hom}(\mathcal{U}, Y)$ the set of such collections.

Let $f : X \to Y$ be a morphism and $\mathcal{U} = \{\sigma_i : U_i \to X\}_i$ a fpqc covering. The compositions $f_i = f \circ \sigma_i : U_i \to X \to Y$ clearly define a morphism $\mathcal{U} \to Y$. This gives us a natural map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathcal{U}, Y)$.

**Theorem 3.9.** *(Grothendieck) Let $X, Y$ be schemes and $\mathcal{U} = \{U_i \to X\}$ an fpqc covering. The natural map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathcal{U}, Y)$ is a bijection. In the standard terminology, this means that a representable functor is a sheaf in the fpqc topology.*

*Proof.* [Vis05, Theorem 2.55]. □

**Definition 3.10.** Let $\mathcal{U} = \{\sigma_i : U_i \to X\}_i$ be a fpqc covering, and call $U_{ij} = U_{ji} = U_i \times_X U_j$, $U_{ijk} = U_i \times_X U_j \times_X U_k$. Consider a collection $(\{f_i\}, \{\eta_{ij}\})$ of affine morphisms $f_i : Y_i \to U_i$ and isomorphisms

$$\eta_{ij} : Y_{ji} = Y_j \times_{U_j} U_{ij} \xrightarrow{\sim} Y_{ij} = Y_i \times_{U_i} U_{ij}$$

such that

$$f_j|_{Y_{ji}} = f_i|_{Y_{ij}} \circ \eta_{ij}.$$

Call $Y_{ijk} = Y_{ij} \times_{U_{ij}} U_{ijk} = Y_i \times_{U_i} U_{ijk}$. We will say that the collection $(\{f_i\}, \{\eta_{ij}\})$ is an *affine morphism with descent data* on $\mathcal{U}$ if, for all triples of indices $i, j, k$, it satisfies the following cocycle condition:

$$\eta_{ik}|_{Y_{kji}} = \eta_{ij}|_{Y_{jki}} \circ \eta_{jk}|_{Y_{kji}}.$$

An arrow between affine morphisms with descent data $\alpha : (\{f_i\}, \{\eta_{ij}\}) \to (\{g_i\}, \{\mu_{ij}\})$ is a collection of commutative diagrams

$$\begin{tikzcd} Y_i \arrow[rr, "\alpha_i"] \arrow[dr, "f_i"'] & & Z_i \arrow[dl, "g_i"] \\ & U_i & \end{tikzcd}$$

such that the following diagram is commutative

$$\begin{array}{ccc} Y_{ji} & \xrightarrow{\alpha_j|_{Y_{ji}}} & Z_{ji} \\ \downarrow{\scriptstyle \eta_{ij}} & & \downarrow{\scriptstyle \mu_{ij}} \\ Y_{ij} & \xrightarrow{\alpha_i|_{Y_{ij}}} & Z_{ij} \end{array}$$

Call $\mathrm{Aff}(X)$ the category of affine morphisms with target $X$, and $\mathrm{Aff}(\mathcal{U})$ the category of affine morphisms with descent data on $\mathcal{U}$.

If $f : Y \to X$ is a morphism of schemes and $\mathcal{U} = \{\sigma_i : U_i \to X\}_i$ is a fpqc covering, $f_i : Y_i = Y \times_X U_i \to U_i$ with the obvious isomorphisms $\eta_{ij} : Y_{ji} \simeq Y \times_X U_{ij} \simeq Y_{ij}$ defines an affine morphism with descent data on $\mathcal{U}$. This defines a functor $\mathrm{Aff}(X) \to \mathrm{Aff}(\mathcal{U})$.

**Theorem 3.11.** *Let $X$ be a scheme and $\mathcal{U} = \{U_i \to X\}$ an fpqc covering. The functor $\mathrm{Aff}(X) \to \mathrm{Aff}(\mathcal{U})$ is an equivalence of categories. In the standard terminology, this means that the fibered category $\mathrm{Aff} \to \mathrm{Sch}$ is a stack in the fpqc topology.*

*Proof.* [Vis05, Theorem 4.33]. □

## 3.2 Torsors

### 3.2.1 Definitions

**Definition 3.12.** A torsor $T$ is a scheme with an action $\alpha : G \times T \to T$ and a $G$-invariant, affine and faithfully flat morphism $\pi : T \to X$ such that $\delta_\alpha = (\mathrm{pr}_T, \alpha) : G \times T \to T \times_X T$ is an isomorphism.

**Example 3.13.** Take an affine group-scheme $G$ and a scheme $X$. Consider the product $T = G \times X$ with the projection $G \times X \to X$ and the action of $G$ on $T$ by left multiplication on itself. Since $\mathrm{pr}_1 \times m : G \times G \to G \times G$ is an isomorphism (using the Yoneda Lemma, $(g, h) \mapsto (g, gh)$ has inverse $(g', h') \mapsto (g', g'^{-1}h')$) then

$$\delta_\alpha : G \times G \times X \to (G \times X) \times_X (G \times X) \simeq G \times G \times X$$

is an isomorphism, too. We will call a torsor $T$ *trivial* if there exists an equivariant isomorphism $T \to G \times X$ over $X$.

**Definition 3.14.** If $T \to X$, $T' \to X$ are respectively a $G$-torsor and a $G'$-torsor, a morphism of torsors is a pair $(f, \psi)$ with

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr] & & T' \arrow[dl] \\ & X & \end{tikzcd}$$

and $\psi : G \to G'$ homomorphism of group-schemes such that the following diagram is commutative:

$$\begin{tikzcd} G \times T \arrow[r, "\alpha"] \arrow[d, "\psi \times f"] & T \arrow[d, "f"] \\ G' \times T' \arrow[r, "\alpha'"] & T' \end{tikzcd}$$

We call $\mathcal{T}(X)$ the category of torsors over $X$.

The definition of a torsor becomes clearer if one thinks of what happens when we work on a base field $k$ and $X$ has a rational point $x_0$: the fiber $T_{x_0}$ is simply a principal homogeneous space for $G$. An alternative way to think to torsors, closer to our intuitive idea of bundle, comes from the fpqc topology.

**Lemma 3.15.** *Let $G$ act on a scheme $T$ and $\pi : T \to X$ be a $G$-invariant morphism. Then, $T$ is a torsor over $X$ if and only if there exists an fpqc covering $\{\sigma_i : U_i \to X\}$ such that $U_i \times_X T \to U_i$ is a trivial torsor for every $i$.*

*Proof.* If $T$ is a torsor, $\{\pi : T_1 \to X\}$ where $T_1$ is a copy of $T$ is an fpqc covering, and the isomorphism $G \times T_1 \simeq T_1 \times_X T$ is $G$-equivariant if $G$ acts trivially on $T_1$.

On the other hand, let $\{U_i \to X\}_i$ be an fpqc covering, and

$$\psi_i : U_i \times_X T \to G \times U_i$$

a $G$-equivariant isomorphism over $U_i$. Thanks to Proposition 3.7, $\pi$ is affine and faithfully flat, and $\delta_\alpha$ is an isomorphism. □

**Example 3.16.** It is rather astonishing that one of the simplest examples of a torsor is given by Galois extensions: the structure of principal bundle remains somehow hidden in the algebraic structure of the fields, until we base change to an fpqc covering where everything is geometrically clearer.

Take a finite Galois extension of fields $L/k$, and consider $\mathrm{Gal}(L/k)$ with the structure of discrete group-scheme. There is an obvious action $\mathrm{Gal}(L/K) \times \operatorname{Spec} L \to \operatorname{Spec} L$, and $\operatorname{Spec} L \to \operatorname{Spec} k$ is $\mathrm{Gal}(L/K)$-invariant, affine and faithfully flat.

By the primitive element theorem, there is an element $\alpha \in L$ generating $L$ as a $k$-algebra. Denote by $f \in k[x]$ its minimal polynomial, we have an isomorphism $L \simeq k[x]/f(x)$. This induces an isomorphism

$$L \otimes L \simeq k[x]/f(x) \otimes L \simeq L[x]/f(x)$$

But $L/k$ is Galois, hence $f$ splits as $\prod_{i=1}^{n}(x - \alpha_i)$ in $L$, where $\alpha = \alpha_1$ and $\alpha_i \neq \alpha_j$ if $i \neq j$. Then $L \otimes L \simeq \prod_i L[x]/(x - \alpha_1) \simeq \prod_i L$. The action of $\mathrm{Gal}(L/K)$ on $L$ permutes the set of roots of $f$, and so $\mathrm{Gal}(L/K) \times \operatorname{Spec} L \to \operatorname{Spec} L \times \operatorname{Spec} L$ is an isomorphism.

**Proposition 3.17.** *Let $T \to X$ be a $G$-torsor, and $Y \to X$ a morphism. Then, $T \times_X Y \to Y$ has a natural structure of $G$-torsor.*

*Proof.* There is an obvious action $G \times T \times_X Y \to T \times_X Y$ induced by the action on $T$. Moreover, $T \times_X Y \to Y$ is faithfully flat and affine because the same is true for $T \to X$. Finally,

$$G \times T \times_X Y \to (T \times_X Y) \times_Y (T \times_X Y) = (T \times_X T) \times_X Y$$

is an isomorphism thanks to the Yoneda Lemma. □

### 3.2.2 Descent data for torsors

A $G$-torsor $\pi : T \to X$ is, in particular, an affine map. Given an fpqc covering $\mathcal{U} = \{\sigma_i : U_i \to X\}$, thanks to Theorem 3.11 giving an affine map on $X$ is equivalent to giving an affine map with descent data on $\mathcal{U}$. We have seen that there exists an fpqc covering where the torsor becomes trivial, now we want to characterize descent data of torsors that are trivial on $\mathcal{U}$.

Let us suppose that $\mathcal{U}$ trivializes $T$. Call $T_i = T \times_X U_i$, $T_{ij} = T_{ji} = T \times_X U_i \times_X U_j$ and $\eta_i : T_i \xrightarrow{\sim} G \times U_i$ the $G$-equivariant trivializations. Then, we have a $G$-equivariant isomorphism

$$\eta_{ij} = \eta_i|_{T_{ij}} \circ \eta_j^{-1}|_{G \times U_{ij}} : G \times U_{ij} \to T_{ij} \to G \times U_{ij}.$$

Call $\varphi_{ji}$ the composition

$$\varphi_{ji} : U_{ij} \xrightarrow{\varepsilon \times \mathrm{id}} G \times U_{ij} \xrightarrow{\eta_{ij}} G \times U_{ij} \xrightarrow{p_1} G.$$

Since $\eta_{ij}$ is $G$-equivariant, it is easy to check that $\eta_{ij}$ is equal to $(g, x) \mapsto (g \cdot \varphi_{ji}(x), x)$ using the Yoneda Lemma.

Using the definition of $\eta_{ij}$, one may check immediately that

$$\eta_{ij}|_{G \times U_{ijk}} \circ \eta_{jk}|_{G \times U_{ijk}} = \eta_{ik}|_{G \times U_{ijk}}.$$

Hence, when we restrict to $U_{ijk}$, we have that

$$(\varphi_{ki}(x), x) = \eta_{ik}(1, x) = \eta_{ij} \circ \eta_{jk}(1, x) =$$

$$= \eta_{ij}(\varphi_{kj}(x), x) = (\varphi_{kj}(x)\varphi_{ji}(x), x)$$

and so $\varphi_{ki} = \varphi_{kj} \cdot \varphi_{ji}$.

**Definition 3.18.** Let $\mathcal{U} = \{U_i \to X\}$ be an fpqc covering such that $\coprod_i U_i \to X$ is quasi-compact. A collection of morphisms $\{\varphi_{ij} : U_{ij} \to G\}$ where $G$ is an affine group-scheme is a *trivial torsor with descent data* on $\mathcal{U}$ if

$$\varphi_{ki} = \varphi_{kj} \cdot \varphi_{ji}$$

for every triple $i, j, k$ when we restrict to $U_{ijk}$.

A *morphism of trivial torsors with descent data* on $\mathcal{U}$

$$(\lambda, \nu) : \{\varphi_{ij} : U_{ij} \to G\} \to \{\varphi'_{ij} : U_{ij} \to G'\}$$

is an homomorphism of group-schemes $\lambda : G \to G'$ together with a family of morphisms $\nu_i : U_i \to G'$ such that

$$\varphi'_{ij} = \nu_j^{-1} \cdot \lambda(\varphi_{ij}) \cdot \nu_i.$$

We call $\mathcal{T}(U)$ the category of trivial torsors with descent data on $\mathcal{U}$.

If $T \to X$ is a torsor trivialized by the fpqc covering $\mathcal{U}$, we have seen above that this defines a trivial torsor with descent data on $\mathcal{U}$. Let $(f, \lambda) : (T, G) \to (T', G')$ be a morphism of torsors trivialized by $\mathcal{U}$, and call $\{\varphi_{ij}\}$, $\{\varphi'_{ij}\}$ their respective descent data. Consider the induced morphisms of torsors $f_i : G \times U_i \to G' \times U_i$, the compositions

$$\nu_i : U_i \xrightarrow{\varepsilon \times \mathrm{id}} G \times U_i \xrightarrow{f_i} G' \times U_i \to G'$$

together with $\lambda : G \to G'$ define a morphism of trivial torsors with descent data. The following commutative diagrams explain why the condition $\lambda(\varphi_{ij}) \cdot \nu_i = \nu_j \cdot \varphi'_{ij}$ is respected:

$$
\begin{tikzcd}
U_{ij} \arrow[dr, "\varepsilon\times\mathrm{id}"] \arrow[drr, bend left, "\nu_j\times\mathrm{id}"] \arrow[ddr, bend right, "\varphi_{ij}\times\mathrm{id}"'] & & \\
& G \times U_{ij} \arrow[r, "f_j"] \arrow[d, "\eta_{ij}"] & G' \times U_{ij} \arrow[d, "\eta'_{ij}"] \\
& G \times U_{ij} \arrow[r, "f_i"] & G' \times U_{ij}
\end{tikzcd}
$$

$$
\begin{tikzcd}
x \arrow[dr, mapsto, "\varepsilon\times\mathrm{id}"] \arrow[drr, mapsto, bend left, "\nu_j\times\mathrm{id}"] \arrow[ddr, mapsto, bend right, "\varphi_{ij}\times\mathrm{id}"'] & & \\
& (\varepsilon, x) \arrow[r, mapsto, "f_j"] \arrow[d, mapsto, "\eta_{ij}"] & (\nu_j(x), x) \arrow[d, mapsto, "\eta'_{ij}"] \\
& (\varphi_{ij}(x), x) \arrow[r, mapsto, "f_i"] & (\lambda(\varphi_{ij}(x))\nu_i(x), x) = (\nu_j(x)\varphi'_{ij}(x), x)
\end{tikzcd}
$$

Call $\mathcal{T}_{\mathcal{U}}(X)$ the category of torsors trivialized by $\mathcal{U}$. We have defined a functor $\mathcal{T}_{\mathcal{U}}(X) \to \mathcal{T}(\mathcal{U})$.

*Remark* 3.19. The trivialization of a torsor on a covering $\mathcal{U}$ is not unique: to define $\mathcal{T}_{\mathcal{U}}(X) \to \mathcal{T}(\mathcal{U})$, we need to choose a trivialization for every torsor. However, this is not really important for our purposes.

**Proposition 3.20.** *Let $\mathcal{U} = \{\sigma_i : U_i \to X\}$ be an fpqc covering such that $\coprod_i U_i \to X$ is quasi-compact. Then $\mathcal{T}_{\mathcal{U}}(X) \to \mathcal{T}(\mathcal{U})$ is an equivalence of categories.*

*Proof.* The proof is just an adaptation of Theorem 3.11. In fact $\mathcal{T}_{\mathcal{U}}(X)$ is a (not full) subcategory of $\mathrm{Aff}(X)$: we will show that $\mathcal{T}(\mathcal{U})$ is a subcategory of $\mathrm{Aff}(\mathcal{U})$, too, and that $\mathcal{T}(\mathcal{U})$ is the essential image of $\mathcal{T}_{\mathcal{U}}(X) \subseteq \mathrm{Aff}(X) \to \mathrm{Aff}(\mathcal{U})$.

Take $\{\varphi_{ij} : U_{ij} \to G\}$ a trivial torsor with descent data. The projections

$$\pi_i : G \times U_i \to U_i$$

together with the maps

$$\eta_{ij} : G \times U_{ij} \to G \times U_{ij}$$

$(g, x) \mapsto (g\varphi_{ij}(x), x)$ define an affine morphism with descent data on $\mathcal{U}$.

Now, if $(\lambda, \nu) : \{\varphi_{ij}\} \to \{\varphi'_{ij}\}$ is a morphism of trivial torsors with descent data, the maps

$$f_i : G \times U_i \to G' \times U_i$$

$(g, x) \mapsto (\lambda(g)\nu_i(x), x)$ define a morphism of descent data. In fact,

$$\eta'_{ij} \circ f_j(g, x) = \eta'_{ij}(\lambda(g)\nu_j(x), x) = (\lambda(g)\nu_j(x)\varphi'_{ij}(x), x) =$$

$$= (\lambda(g\varphi_{ij}(x))\nu_i(x), x) = f_i(g\varphi_{ij}(x), x) = f_i \circ \eta_{ij}(g, x).$$

The morphism of descent data $\{f_i\}$ is clearly equivariant with respect to $\lambda : G \to G'$.

On the other hand, take a morphism of descent data

$$\{f_i : G \times U_i \to G' \times U_i\}$$

equivariant with respect to $\lambda : G \to G'$. Call $\nu_i$ the composition

$$U_i = \operatorname{Spec} k \times U_i \xrightarrow{\varepsilon \times \mathrm{id}} G \times U_i \xrightarrow{f_i} G' \times U_i \to G',$$

we have that $f_i(g, x) = \lambda(g)\nu(x)$ on $G \times U_i$ because $f_i$ is $G$-equivariant. Since $\{f_i\}$ is an morphism of descent data, we have the equality $f_i \circ \eta_{ij} = \eta'_{ij} \circ f_j$ on $G \times U_{ij}$. Evaluating it on $(\varepsilon, x)$, we get

$$f_i \circ \eta_{ij}(\varepsilon, x) = f_i(\varphi_{ij}(x), x) = (\lambda(\varphi_{ij}(x))\nu_i(x), x) =$$

$$= \eta'_{ij} \circ f_j(\varepsilon, x) = \eta'_{ij}(\nu_j(x), x) = (\nu_j(x)\varphi'_{ij}(x), x)$$

and hence $\lambda(\varphi_{ij})\nu_i = \nu_i\varphi'_{ij}$, $(\lambda, \nu)$ is a morphism of trivial torsors with descent data.

To sum up, we have identified $\mathcal{T}(\mathcal{U})$ with the subcategory of $\operatorname{Aff}(U)$ whose objects are collection of the form $\{G \times U_i\}$ for some $G$ with $G$-equivariant maps $\eta_{ij} : G \times U_{ij} \to G \times U_{ij}$, and whose arrows are $G$-equivariant, and we have seen that $\mathcal{T}_{\mathcal{U}}(X) \to \mathcal{T}(\mathcal{U})$ is fully faithful. To show that it is also essentially surjective, we must check that the affine map $T \to X$ given by such descent data defines a torsor. We must give an action $\alpha : G \times T \to T$, show that $T \to X$ is faithfully flat and $G$-equivariant and that $\delta_\alpha : G \times T \to T \times_X T$ is a isomorphism. These facts are all trivial at the level of descent data, then we may apply Proposition 3.7. □

*Remark* 3.21. We have asked the covering to be such that $\coprod_i U_i \to X$ is a quasi-compact morphism: we need to do this in order to use Proposition 3.7. Anyway, this is not a problem: since a torsor $T \to X$ is trivial when restricted to the covering $\{T \to X\}$ and $T \to X$ is affine, we can always suppose $\coprod_i U_i \to X$ to be quasi-compact.

### 3.2.3 Induced torsor

**Proposition 3.22** (Induced torsor). *Let $T_0 \to X$ be a $G_0$-torsor, where $G_0$ is an affine group-scheme. Call* $\mathrm{Hom}(G_0, -)$ *the category of homomorphisms of affine group-schemes $G_0 \to -$, where an arrow is a commutative diagram*

$$\begin{tikzcd} & G_0 \arrow[dl] \arrow[dr] & \\ G \arrow[rr] & & G' \end{tikzcd}$$

*Similarly, call* $\mathrm{Hom}(T_0, -)$ *the category of morphisms of torsors $T_0 \to -$. Then, there exists a functor $\mathcal{I}_{T_0} : \mathrm{Hom}(G_0, -) \to \mathrm{Hom}(T_0, -)$ such that the composition with the forgetful functor*

$$\mathrm{Hom}(G_0, -) \xrightarrow{\mathcal{I}_{T_0}} \mathrm{Hom}(T_0, -) \to \mathrm{Hom}(G_0, -)$$

*is the identity.*

*Proof.* Let $\mathcal{U} = \{U_i \to X\}$ be an fpqc covering trivializing $T_0$ such that $\coprod_i U_i \to X$ is quasi-compact and $\varphi_{ji} : U_{ij} \to G_0$ are the morphisms giving descent data for $T_0$ as in Proposition 3.20. If $\psi : G_0 \to G$ is an homomorphism of affine schemes, the compositions

$$\psi \circ \varphi_{ji} : U_{ij} \to G_0 \to G$$

clearly satisfy the cocycle condition on $U_{ijk}$ because $\psi$ is an homomorphism:

$$\begin{aligned} \psi \circ \varphi_{ki} = \psi \circ (\varphi_{kj} \cdot \varphi_{ji}) = \\ = (\psi \circ \varphi_{kj}) \cdot (\psi \circ \varphi_{ji}). \end{aligned}$$

Hence, we have descent data defining a $G$-torsor $T$. The map of descent data $(\psi, \nu)$, where $\nu_i : U_i \to G$ is the constant morphism on the identity, descends to a morphism $f : T_0 \to T$ defining a morphism of torsors $(f, \psi)$.

If $\psi' : G_0 \to G'$ induces a morphism of torsors $f' : T_0 \to T'$ and $\lambda : G \to G'$ is an homomorphism such that $\psi' = \lambda \circ \psi$, we have a map of descent data $(\lambda, \mu)$, with $\mu_i : U_i \to G'$ constant morphism on the identity, descending to a morphism $T \to T'$ such that

$$\begin{tikzcd} & T_0 \arrow[dl] \arrow[dr] & \\ T \arrow[rr] & & T' \end{tikzcd}$$

is commutative.

We want now to show that the construction of $\mathcal{I}_{T_0}$ does not depend on the choosing of the covering $\mathcal{U}$. Let $\mathcal{U}'$ be another fpqc covering trivializing $T$, then $\mathcal{U} \sqcup \mathcal{U}'$ is a third fpqc covering trivializing $T$. Consider now $T'$ the torsor induced by $\psi$ using $\mathcal{U}'$, and $T''$ the one using $\mathcal{U} \sqcup \mathcal{U}'$: clearly the descent data of $T$ and $T''$ coincide on $\mathcal{U}$, which is a covering, hence this defines a unique isomorphism $T \simeq T''$. For the same reason, $T'' \simeq T'$, and these isomorphisms are functorial. □

**Corollary 3.23.** *Let $\mathcal{F} : \mathcal{J} \to \mathrm{AffGrp}_k$ be a diagram $j \mapsto G_j$, $(G_0, \psi_0)$ an universal cone for $\mathcal{F}$ and $T_0 \to X$ a $G_0$-torsor. Then, there exists a functor $\mathcal{F}_{\psi_0} : \mathcal{J} \to \mathcal{T}(X)$ with limit $T_0$ such that $\mathcal{F}$ is the composition of $\mathcal{F}_{\psi_0}$ and the forgetful functor $\mathcal{T}(X) \to \mathrm{AffGrp}_k$.*

*Proof.* A cone $(G_0, \psi_0)$ for $\mathcal{F}$ can be thought as a functor $\psi_0 : \mathcal{J} \to \mathrm{Hom}(G_0, -)$ such that $\psi_0$ composed with the forgetful functor $\mathrm{Hom}(G_0, -) \to \mathrm{AffGrp}_k$ is $\mathcal{F}$. Now, $\mathcal{I}_{T_0} \circ \psi_0 : \mathcal{J} \to \mathrm{Hom}(T_0, -)$ is a cone $(T_0, (f_0, \psi_0))$ for the composition

$$\mathcal{F}_{\psi_0} : \mathcal{J} \xrightarrow{\psi_0} \mathrm{Hom}(G_0, -) \xrightarrow{\mathcal{I}_{T_0}} \mathrm{Hom}(T_0, -) \to \mathcal{T}(X).$$

We only need to show that $(T_0, (f_0, \psi_0))$ is universal. Let $(T, (f, \psi))$ be another cone for $\mathcal{F}_{\psi_0}$, where $T$ is a $G$-torsor. Let $\mathcal{U} : \{U_i \to X\}$ be an fpqc covering trivializing $T_0$ such that $\coprod_i U_i \to X$ is quasi-compact, the one used to construct $\mathcal{I}_{T_0}$. Since we have seen that the construction of $\mathcal{I}_{T_0}$ does not depend on $\mathcal{U}$, we may take an opportune refinement trivializing $T$, too. Clearly, the idea is to define $h : T \to T_0$ at the level of descent data, but we must pay some attention.

Since $\mathcal{I}_{T_0}$ does not depend on the covering, we may refine $\mathcal{U}$ and suppose that it trivializes $T$, too. Let $\{\varphi_{ij} : U_{ij} \to G\}$, $\{\varphi_{0,ij} : U_{ij} \to G_0\}$

be respectively the descent data of $T$ and $T_0$. At the level of descent data, the cone $(T,(f,\psi))$ is a collection of pairs $(\psi_l,\nu_l)$, with $\psi_l : G \to G_l$ and $\nu_{l,i} : U_i \to G_l$. The fact that $(G_0,\psi)$ is universal gives us unique morphisms $\lambda : G \to G_0$ and $\nu_i : U_i \to G_0$ such that $\psi_l = \psi_{0,l} \circ \lambda$ and $\nu_{l,i} = \psi_{0,l} \circ \nu_i$. Finally, $(\lambda,\nu)$ gives us the desired unique morphism of cones $(T,\psi) \to (T_0,\psi_0)$. $\square$

## 3.3 Equivariant sheaves

Given a scheme $X$, call $\mathrm{QCoh}(X)$ the category of quasi-coherent sheaves over $X$. If we have two quasi-coherent sheaves $\xi,\sigma$ over two schemes $X,Y$, we say that a morphism $\xi \to \sigma$ is a pair $(f,g)$ where $f : X \to Y$ is a morphism of schemes and $g : \xi \to f^*\sigma$ is a morphism of quasi-coherent sheaves over $X$. Call QCoh the category of pairs $(X,\xi)$ with $\xi$ quasi-coherent sheaf over $X$. With abuse of notation, we will indicate morphisms $\xi \to \sigma$ as "commutative diagrams"

$$\begin{tikzcd} \xi \arrow[r] \arrow[d, mapsto] & \sigma \arrow[d, mapsto] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

*Remark* 3.24. The careful reader may have noticed a problem in our definition of QCoh. If $(Z,\lambda)$ is a third object of QCoh and $(f',g') : (Y,\sigma) \to (Z,\lambda)$ is another morphism, we would like to define the composition $(f',g') \circ (f,g) : (X,\xi) \to (Z,\lambda)$ as a pair $(f'f,g'')$ with $g'' : \xi \to (f'f)^*\lambda$. But when we try to compose $g$ and $g'$ we get a morphism of sheaves $\xi \to f^*\sigma \to f^*f'^*\lambda$, and $(f'f)^*\lambda$ is different from $f^*f'^*\lambda$: they are not equal, they are only isomorphic. Taking into account this isomorphism complicates a lot the treatment without real advantages apart from rigour. Here, we will simply ignore the problem, identifying $(f'f)^*\lambda$ and $f^*f'^*\lambda$. An explanation of why we can do this is contained in [Vis05, sect. 3.2.1].

### 3.3.1 Definitions

Given an affine group-scheme $G$ and a quasi-coherent sheaf $\lambda$ on a scheme $T$ with an action $\alpha : G \times T \to T$, we want to define what an action of $G$ on $\lambda$ should be. The intuitive idea is that $G$ should act somehow on the pair $(T,\lambda)$. In our categorical language, it is clear what we must do: if $X \to T$ is a "$X$-point" of $T$ and $\xi$ is a section over $X$, i.e. a quasi coherent sheaf over $X$ with a commutative diagram

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & \lambda \arrow[d] \\ X \arrow[r] & T \end{tikzcd}$$

the points of $G(X)$ should act on the pair $(X, \xi)$ compatibly with the action on $T$.

**Definition 3.25.** A $G$-equivariant sheaf over $T$ is a sheaf $\lambda$ together with an action of $G(X)$ on the set $\mathrm{Hom}(\xi, \lambda)$ for any quasi-coherent sheaf $\xi$ over $X$, such that the following two conditions are satisfied.

1. For any morphism of quasi-coherent sheaves $\varphi : \eta \to \xi$ over a morphism of schemes $Y \to X$, the induced function $\varphi^* : \mathrm{Hom}(\xi, \lambda) \to \mathrm{Hom}(\eta, \lambda)$ is equivariant with respect to the group homomorphism $G(X) \to G(Y)$.

2. The obvious induced function $\mathrm{Hom}(\xi, \lambda) \to \mathrm{Hom}(X, T)$ is $G(X)$ equivariant for every $X$ and every $\xi$ over $X$.

**Example 3.26.** The structure sheaf of a scheme $T$ with an action of $G$ is a $G$-equivariant sheaf in a natural way. Consider a commutative diagram

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & \mathcal{O}_T \arrow[d] \\ X \arrow[r, "f"] & T \end{tikzcd}$$

which is a morphism of quasi-coherent sheaves $\varphi : \xi \to f^*\mathcal{O}_T = \mathcal{O}_X$ over $X$.

Now, take $g \in G(X)$, the action of $g$ on $\xi \to \mathcal{O}_T$ is defined by the composition

$$\begin{tikzcd} \xi \arrow[r, "\varphi"] \arrow[dr] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_T \arrow[d] \\ & X \arrow[r, "gf"] & T \end{tikzcd}$$

**Example 3.27.** We now want to refine the example above: take a representation of $G$ on $V$, and consider the free sheaf $V \otimes \mathcal{O}_T$ over $T$. When $V = k$ is the trivial representation, $k \otimes \mathcal{O}_T \simeq \mathcal{O}_T$ is simply the structure sheaf.

Roughly speaking, we want to define a structure of equivariant sheaf on $V \otimes \mathcal{O}_T$ such that $G$ acts both on $V$ and $\mathcal{O}_T$.

Consider a commutative diagram

$$\begin{tikzcd}
\xi \arrow[r] \arrow[d, maps to] & V \otimes \mathcal{O}_T \arrow[d, maps to] \\
X \arrow[r, "f"] & T
\end{tikzcd}$$

which is a morphism of quasi-coherent sheaves $\varphi : \xi \to f^*(V \otimes \mathcal{O}_T) = V \otimes \mathcal{O}_X$. Take $g \in G(X)$, we want to define the action of $g$ on $\varphi$.

Note that $g$ defines a morphism $V \otimes \mathcal{O}_X \to V \otimes \mathcal{O}_X$ of $\mathcal{O}_X$ sheaves: if $U \subseteq X$ is an open subset, $g|_U$ defines an $\mathcal{O}_X(U)$ linear map $V \otimes \mathcal{O}_X(U) \to V \otimes \mathcal{O}_X(U)$. Call $\rho(g)$ this morphism.

Now, split the diagram above as

$$\begin{tikzcd}
\xi \arrow[r, "\varphi"] \arrow[dr, maps to] & V \otimes \mathcal{O}_X \arrow[r, "\mathrm{id}"] \arrow[d, maps to] & V \otimes \mathcal{O}_X \arrow[r, "\mathrm{id} \otimes f^{\#}"] \arrow[dl, maps to] & V \otimes \mathcal{O}_T \arrow[d, maps to] \\
& X \arrow[rr, "f"] & & T
\end{tikzcd}$$

and define the action of $g$ on $\varphi$ as the composition

$$\begin{tikzcd}
\xi \arrow[r, "\varphi"] \arrow[dr, maps to] & V \otimes \mathcal{O}_X \arrow[r, "\rho(g)"] \arrow[d, maps to] & V \otimes \mathcal{O}_X \arrow[r, "\mathrm{id} \otimes g f^{\#}"] \arrow[dl, maps to] & V \otimes \mathcal{O}_T \arrow[d, maps to] \\
& X \arrow[rr, "gf"] & & T
\end{tikzcd}$$

Roughly speaking, $g$ acts on a section $v \otimes s$ as $gv \otimes gs$.

*Remark* 3.28. Classically, one defines an equivariant sheaf $\lambda$ on $T$ giving an isomorphism of sheaves $\varphi : \mathrm{pr}_2^* \lambda \simeq \alpha^* \lambda$ on $G \times T$ which satisfies a compatibility condition on $G \times G \times T$. Call $\alpha_0 : G \times G \times T \to T$ the natural projection, $\alpha_1 = \alpha \circ \mathrm{pr}_{23}$ the multiplication of the second and third factors, and $\alpha_2 = \alpha \circ (\mathrm{id}_G \times \alpha)$ the full multiplication. Then following diagram of sheaves on $G \times G \times T$ must be commutative:

$$\begin{tikzcd}
\alpha_0^*\lambda \arrow[rr, "(m_G\times \mathrm{id}_T)^*\varphi"] \arrow[dr, "\mathrm{pr}_{23}^*\,\varphi"'] & & \alpha_2^*\lambda \\
& \alpha_1^*\lambda \arrow[ur, "(\mathrm{id}_G\times\alpha)^*\varphi"'] &
\end{tikzcd}$$

The equivalence of our definition with the classical one is shown in [Vis05, Proposition 3.49].

### 3.3.2 Equivariant sheaves on torsors

The reason why we are interested in $G$-equivariant sheaves is the following result of descent theory.

**Theorem 3.29.** *Let $\pi : T \to X$ a $G$-torsor, $\mathrm{QCoh}(X)$ the category of quasi-coherent sheaves on $X$ and $\mathrm{QCoh}^G(T)$ the category of quasi-coherent $G$-equivariant sheaves on $T$. Then $\pi^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}^G(T)$ is an equivalence of categories.*

*Proof.* [Vis05, Theorem 4.46]. □

**Corollary 3.30.** *Let $\pi : T \to X$ be a $G$-torsor and $\mathrm{Vect}(X) \subseteq \mathrm{QCoh}(X)$ the category of vector bundles on $X$, i.e. the category of quasi-coherent locally free sheaves of finite rank. Then, $\pi^* : \mathrm{Vect}(X) \to \mathrm{Vect}^G(T)$ is an equivalence of categories.*

*Proof.* Thanks to [EGAIV-2, Proposition 2.5.2], a quasi-coherent sheaf $E$ on $X$ is a vector bundle of rank $r$ if and only if the same is true for $\pi * E$ on $T$. Then, apply Theorem 3.29. □

## 3.4 Quotients

Given an action of $G$ on $X$, we would like to define a quotient $G\backslash X$. Unfortunately, quotients do not always exist as they do for classical groups acting on sets. It is easy to show their existence at the level of functors, but the problem is that in general this functor will not be representable. There is an entire theory studying the existence of quotients: here we restrict ourselves to some special cases, referring the reader to [MFK94] for a more general treatment.

### 3.4.1 Geometric and categorical quotients

**Definition 3.31.** Let $G$ be a group-scheme acting on the left on $X$. An invariant morphism $q : X \to Y$ is a *categorical quotient* if, for every other $G$-invariant morphism $X \to Z$, there exists a unique morphism $Y \to Z$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd}$$

The quotient is often written as $G\backslash X$. If the action is on the right, $X/G$.

**Definition 3.32.** Given an action $\alpha : G \times X \to X$ and a $j$-invariant morphism $f : X \to Y$, we will say that the action is *transitive on the fibers of $f$* if the map $G \times X \to X \times_Y X$ defined by $(g, x) \mapsto (gx, x)$ using the Yoneda Lemma is surjective. If $f : X \to \operatorname{Spec} k$ is simply the structure morphism, we will say that the action is *transitive*.

**Definition 3.33.** Let $\alpha : G \times X \to X$ be an action and $p \in X$ a set-theoretical point. We call the *orbit set* of $p$ the subset of $X$

$$Gp = \alpha(\mathrm{pr}_X^{-1}(p)) = \mathrm{pr}_X(\alpha^{-1}(p)).$$

Call $(X/G)_{\mathrm{rs}}$ the space of the orbit sets with the quotient topology. The $G$-invariant sections of $\mathcal{O}_X$ define a sheaf of $k$-algebras $\mathcal{O}_X^G$ on $(X/G)_{\mathrm{rs}}$, and the canonical projection $X \to (X/G)_{\mathrm{rs}}$ is a morphism of ringed spaces.

**Lemma 3.34.** *Let $f : X \to Y$ be a $G$-invariant map. If the action is transitive on the fibers of $f$, then $f$ separates orbit sets.*

*Proof.* Let $x_1, x_2$ be two points of $X$ such that $f(x_1) = f(x_2)$ and $K$ a field extension of $k$ containing both $k(x_1)$ and $k(x_2)$, then $(x_1, x_2)$ defines a $K$-rational point of $X \times_Y X$. Since $G \times X \to X \times_Y X$ is surjective, a point $p$ over $(x_1, x_2)$ shows that $x_1$ and $x_2$ are contained in the same orbit set. □

**Example 3.35.** The contrary is not true. Let $G = \operatorname{Spec} k$ be the trivial group-scheme acting trivially on $X = \operatorname{Spec} L$ where $L/k$ is a field extension and $f : X \to Y = \operatorname{Spec} k$ the structure morphism. Since the action is trivial, $f$ is $G$-invariant, and clearly it separates orbit sets. Now, the map

$$G \times X = X \to X \times_Y X = \operatorname{Spec} L \otimes_k L$$

will not be, in general, surjective, for example for $k = \mathbb{R}$ and $L = \mathbb{C}$.

**Definition 3.36.** Let $G$ be a group-scheme acting on $X$. An invariant morphism $q : X \to Y$ is a *geometric quotient* if:

- $q$ is surjective,
- the action is transitive on the fibers of $q$,
- $q$ is *submersive*, i.e. $U \subseteq Y$ is open if and only if $q^{-1}(U)$ is open,
- $\mathcal{O}_Y \subseteq q_*\mathcal{O}_X$ is the subsheaf of invariant sections.

**Proposition 3.37.** *If $Y = (X/G)_{\mathrm{rs}}$ is a scheme, $X \to Y$ is a categorical quotient.*

*Proof.* Let $\alpha : G \times X \to X$ be the action and $f : X \to Z$ a $G$-invariant morphism. Take $\{V_i\}_i$ an affine covering of $Z$, then $f^{-1}(V_i) \subseteq X$ is a $G$-invariant open set. Set theoretically, call $U_i = q(f^{-1}(V_i)) \subseteq Y$: we have that $q^{-1}(U_i) = f^{-1}(V_i)$ because $(X/G)_{\mathrm{rs}}$ is the space of the orbits.

Since $(X/G)_{\mathrm{rs}}$ has the quotient topology and $q^{-1}(U_i) = f^{-1}(V_i)$ is open, we have that $U_i$ is open, too. Moreover, since $q$ is surjective and $\{f^{-1}(V_i)\}_i$ is a covering of $X$, $\{U_i\}_i$ is a covering of $Y$, too. Now, $f^{\#}|_{V_i} : \mathcal{O}_Z(V_i) \to \mathcal{O}_X(q^{-1}(U_i))$ has image contained in $\mathcal{O}_X(q^{-1}(U_i))^G = \mathcal{O}_Y(U_i)$, hence we have defined a unique morphism $f'_i : U_i \to V_i$ (because $V_i$ is affine). Passing to an opportune refinement of $\{V_i\}$, uniqueness also implies that $f'_i|_{U_i \cap U_j} = f'_j|_{U_i \cap U_j}$, hence the morphisms $f'_i$ glue as $f' : Y \to Z$. □

**Corollary 3.38.** *Geometric quotients are categorical quotients.*

*Proof.* Let $q : X \to Y$ be a geometric quotient. The projection $q$ separates orbit sets, is surjective and submersive, and hence gives an homeomorphism of $Y$ with $(X/G)_{\mathrm{rs}}$. Moreover, their structure sheaves are both $\mathcal{O}_X^G$, and hence $Y \simeq (X/G)_{\mathrm{rs}}$. □

Clearly, categorical quotients are unique up to a unique isomorphism. Since geometric quotients are also categorical quotients, they are unique, too.

### 3.4.2 Existence theorems

**Lemma 3.39.** *If $\pi : T \to X$ is a $G$-torsor, $X$ is a geometric quotient of $T$ by the action of $G$.*

*Proof.* The fact that $G \times T \to T \times_X T$ is an isomorphism ensures that the action is transitive on the fibers of $\pi$, and Lemma 3.4 implies that $\pi$ is submersive and clearly $\pi$ is surjective. The only non trivial fact we need to check is that $\mathcal{O}_X \subseteq \pi_* \mathcal{O}_T$ is the subsheaf of $G$-invariant sections.

Clearly, $\mathcal{O}_X \subseteq \pi_* \mathcal{O}_T^G$. On the other hand, let $s \in \pi_* \mathcal{O}_T(U) = \mathcal{O}_T(\pi^{-1}(U))$ be $G$-invariant, with $U \subseteq X$ open subscheme. This means that the pullbacks

$$p_2^\#(s), \alpha^\#(s) \in \mathrm{H}^0(G \times \pi^{-1}(U)) \simeq \mathrm{H}^0(\pi^{-1}(U) \times_U \pi^{-1}(U))$$

are equal, and these are precisely the two restriction of $s$ from the two components on the "intersection" $\pi^{-1}(U) \times_U \pi^{-1}(U)$. Considering the section $s$ and its pullbacks as morphisms to $\mathbb{A}^1$, Theorem 3.9 implies that $s$ descends to a morphism $U \to \mathbb{A}^1$: this means exactly that $s$ is the pullback of some section of $\mathcal{O}_X(U)$.

□

**Definition 3.40.** Let $G$ be an affine group-scheme with an action $\alpha : G \times X \to X$, $F$ a field and $x$ a point in $X(F)$. The *stabilizer* $G_x$ of $x$ is the subgroup functor of $G \times \operatorname{Spec} F$ defined by

$$G_x(S) = \{g \in G \times \operatorname{Spec} F(S) \mid g \cdot x = x\}.$$

**Lemma 3.41.** *$G_x$ is represented by a closed subgroup of $G \times \operatorname{Spec} F$.*

*Proof.* Let $G'_x$ be defined by the following cartesian diagram:

$$\begin{array}{ccc}
G'_x & \longrightarrow & G \times \operatorname{Spec} F \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}_G \times x \times \mathrm{id}} \\
 & & G \times X \times \operatorname{Spec} F \\
 & & \Big\downarrow{\scriptstyle \alpha \times \mathrm{id}} \\
\operatorname{Spec} F & \xrightarrow{x \times \mathrm{id}} & X \times \operatorname{Spec} F
\end{array}$$

It is easy to check that $G'_x$ represents $G_x$, and $G_x \to G \times \operatorname{Spec} F$ is clearly a morphism of group-schemes which is a closed embedding thanks to Lemma 2.35. □

**Proposition 3.42.** *Let $\alpha : G \times X \to X$ be an action and $p \in X$ a rational point.*

*(i) If $G$ and $X$ are of finite type, the orbit set $Gp$ has a natural structure of reduced scheme of finite type with a morphism $Gp \to X$, and $Gp$ is open in $\overline{Gp}$.*

(ii) *If, moreover, $G$ and $X$ are geometrically reduced and $G$ is connected, we have a faithfully flat morphism $\alpha_p : G = G \times \operatorname{Spec} k \to Gp$.*

(iii) *$Gp$ has dimension $\dim G - \dim G_p$ at $p$.*

(iv) *If, moreover, $Gp$ is contained in an affine open subset $U \subseteq X$, $\alpha_p : G \to Gp$ is a $G_p$-torsor with respect to multiplication on the right, and hence $Gp = G/G_p$ as a geometric quotient.*

*Proof.* (i) Consider $Gp$ the orbit set of $p$, which is the set theoretical image of

$$G \times \operatorname{Spec} k \xrightarrow{\mathrm{id} \times p} G \times X \xrightarrow{\alpha} X.$$

We want to show that $Gp$ is locally closed. Since vertical arrows in the diagram

$$\begin{array}{ccc} G_{\bar{k}} & \longrightarrow & X_{\bar{k}} \\ \downarrow & & \downarrow \\ G & \longrightarrow & X \end{array}$$

are surjective and submersive, it is enough to show the thesis when $k = \bar{k}$. Thanks to [Har77, Exercise II.3.19], $Gp$ is constructible, and hence contains a nonempty open subset $U$ of $\overline{Gp}$. Then, $U' = \bigcup_{g \in G(k)} gU$ is open in $\overline{Gp}$ and contains every closed point of $Gp$ thanks to Nullstellensatz, hence $Gp = U'$ is locally closed.

Since $Gp$ is locally closed, there is a natural structure of reduced scheme of finite type on $Gp$ induced by the one of $X$. We call $Gp$ with this structure the *orbit* of $p$.

(ii) Since $G$ is geometrically reduced, we have a surjective map of reduced schemes $\alpha_p : G = G \times \operatorname{Spec} k \to Gp$. Consider the following cartesian diagram:

$$\begin{array}{ccccccc} G_{\bar{k}} & \longrightarrow & Gp_{\bar{k}} & \longrightarrow & X_{\bar{k}} & \longrightarrow & \operatorname{Spec} \bar{k} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G & \longrightarrow & Gp & \longrightarrow & X & \longrightarrow & \operatorname{Spec} k \end{array}$$

Since $Gp \to X$ is a locally closed subscheme, $Gp_{\bar{k}} \to X_{\bar{k}}$ is a locally closed subscheme, too. Set theoretically, it coincides with $G_{\bar{k}}p$, and both are reduced because $X$ is geometrically reduced, hence $G_{\bar{k}}p = Gp\bar{k}$. Hence, thanks to Proposition 3.7, to prove that $G \to Gp$ is faithfully flat we may suppose $k = \bar{k}$.

We know that $Gp$ is integral because it is the image of $G$ and it is reduced. Thanks to generic flatness [EGAIV-2, Théorème 6.9.1], there is an open nonempty subscheme $U \subseteq Gp$ such that $\alpha_p$ is flat when restricted to $\alpha_p^{-1}(U)$. Now take a closed point $g \in G$ and, thanks to the fact that $G$ is Jacobson ([Bou64, V.3.4, Theorem 3]), a closed point $g' \in \alpha_{x_0}^{-1}(U)$. Thanks to Nullstellensatz, both $g$ and $g'$ are rational and multiplication by $g'g^{-1}$ gives automorphisms of $G$ and $Gp$ that we both call $\psi$ with abuse of notation. Since $\alpha_p$ is $G$-equivariant,

$$\alpha_p(g) = \psi^{-1} \circ \alpha_p \circ \psi(g) = \psi^{-1} \circ \alpha_p(g')$$

and hence $\alpha_p$ is flat at $g$.

This shows that $\alpha_p$ is flat at all closed points: but then, thanks to [EGAIV-3, Théorème 11.3.1], $\alpha_p$ is flat everywhere.

(iii) This is an immediate consequence of [EGAIV-2, Corollaire 6.1.2].

(iv) The fact that $Gp$ is contained in the affine open subset $U$ implies that $G \to Gp$ is an affine morphism. In fact, $Gp$ is an open subset of an affine, closed subscheme $\overline{Gp} \cap U \subseteq U$, and the property of being affine is local in the codomain. We already know that $\alpha_p$ is faithfully flat, we need only to show that $G \times G_p \to G \times_{Gp} G$ defined by $(g, h) \mapsto (gh, g)$ using the Yoneda Lemma is an isomorphism, but this is obvious by definition of stabilizer.

□

We want now to prove the existence of geometric quotients when $G$ is a finite group-scheme and the orbits are contained in open affine subsets. In order to do this, we need a technical lemma.

**Lemma 3.43.** *Let $A$ be a finite $R$-algebra which is free as an $R$-module, and $f : \operatorname{Spec} A \to \operatorname{Spec} R$ the induced map. Then,*

$$f(V(a)) = V(\operatorname{Norm}(a))$$

*for every $a \in A$, where* $\operatorname{Norm} : A \to R$ *is the map sending $a \in A$ to the determinant of $\cdot a : A \to A$.*

*Proof.* Take $p \in \operatorname{Spec} R$ and $f^{-1}(p) = \{q_1, \dots, q_n\}$. Since $R \to A$ is injective and finite, $f$ is surjective, $n \geq 1$. Our claim is $a \in \bigcup_i q_i \iff \operatorname{Norm}(a) \in p$.

Now, since by definition $\operatorname{Norm}(a)$ is the determinant of the $R$-linear map $\cdot a : A \to A$, $\operatorname{Norm}(a) \notin p$ if and only if $\cdot a : A_p \to A_p$ is invertible, i.e. if and only if $a \in A_p^*$. Finally, this is exactly like asking $a$ not to be in $\bigcup_i q_i$. □

**Definition 3.44.** Let $\alpha : G \times X \to X$ be an action. We will say that the action is *free* if

$$(\alpha, \mathrm{p}_X) : G \times X \to X \times X$$

is a closed immersion.

**Theorem 3.45.** *Let $G = \operatorname{Spec} A$ be a finite group-scheme and $\alpha : G \times X \to X$ an action such that the orbit set of any point is contained in an affine open subset of $X$.*

*(i) $Y = (X/G)_{\mathrm{rs}}$ is a scheme, and hence $X \to Y$ is a categorical quotient. If $X$ is affine, $Y = \operatorname{Spec} \mathrm{H}^0(X, \mathcal{O})^G$ is affine, too.*

*(ii) If the action is free, then $\pi$ is flat of degree $n$, i.e. $\pi_* \mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module of rank $n$ where $n = \dim_k A$, and*

$$(\alpha, \mathrm{p}_X) : G \times X \to X \times_Y X$$

*is an isomorphism. Hence, $\pi : X \to Y$ is a torsor.*

*Proof.* Let us prove point (i).

*Step 1*: reduction to the affine case.

Let us suppose that we have shown that $(X/G)_{\mathrm{rs}}$ is a scheme when $X$ is affine, we claim that this implies the general case. Clearly, it is enough to show that we may cover $X$ with affine and $G$-invariant open subschemes.

Hence, fix $p \in X$ and consider an affine open subset $U \subseteq X$ containing $Gp$. Consider $U_0 \subseteq U$ the maximal $G$-invariant subset of $U$, we claim that it is open. In fact

$$U_0 = X \setminus \bigcup_{q \in X \setminus U} Gq = X \setminus \mathrm{pr}_X(\alpha^{-1}(X \setminus U))$$

is open because $G$ is finite and hence $\mathrm{pr}_X : G \times X \to X$ is proper. Since $Gp$ is finite, there exists $f \in \mathrm{H}^0(U)$ such that $Gp \subseteq U_f \subseteq U_0$ [AM69, Proposition 1.11]. As before, let $U_{f,0} \subseteq U_f$ the maximal $G$-invariant open subset, we claim that $U_{f,0}$ is affine.

Let $g \in \mathrm{H}^0(G \times U_0) = \mathrm{H}^0(G) \otimes \mathrm{H}^0(U_0)$ be the pullback of $f|_{U_0}$ by $\alpha : G \times U_0 \to U_0$. Now consider $\mathrm{H}^0(G) \otimes \mathrm{H}^0(U_0)$ as a free $\mathrm{H}^0(U_0)$ module with respect to the standard immersion $\mathrm{H}^0(U_0) \to \mathrm{H}^0(G) \otimes \mathrm{H}^0(U_0)$, which defines the projection $\mathrm{pr}_{U_0} : G \times U_0 \to U_0$. Using Lemma 3.43, we get that $\mathrm{Norm}(g) \in \mathrm{H}^0(U_0)$ is different from 0 on $q$ if and only if $G(q) \subseteq U_f$, and hence $U_{f\,\mathrm{Norm}(g)} = U_{f,0}$ is affine.

Now, consider the affine case, $X = \operatorname{Spec} B$, and let $\rho : B \to B \otimes A$ be the comodule defining the action $\alpha : G \times X \to X$, and $j : B \to B \otimes A$ the map $b \mapsto b \otimes 1$. Define the subring of $G$-invariants $C := B^G \subseteq B$ as

$$\{b \in B | \sigma(b) = j(b)\},$$

we claim that $Y = \operatorname{Spec} C$ is isomorphic to $(X/G)_{\mathrm{rs}}$.

*Step 2*: $B$ is integral over $C$.

For $b \in B$, multiplication by $\rho(b)$ is an endomorphism of $B \otimes A$ with characteristic polynomial

$$\chi(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0 \in B[t].$$

We claim that $\chi(t) \in B^G[t]$ and $\chi(b) = 0$. The diagrams

$$\begin{array}{ccc} B \otimes A & \xrightarrow{\mathrm{id} \otimes m} & B \otimes A \otimes A \\ {\scriptstyle j}\big\uparrow & & {\scriptstyle j_{1,2}}\big\uparrow \\ B & \xrightarrow{\quad j \quad} & B \otimes A \end{array} \quad \text{and} \quad \begin{array}{ccc} B \otimes A & \xrightarrow{\rho \otimes \mathrm{id}} & B \otimes A \otimes A \\ {\scriptstyle j}\big\uparrow & & {\scriptstyle j_{1,2}}\big\uparrow \\ B & \xrightarrow{\quad \rho \quad} & B \otimes A \end{array}$$

are cartesian, where $j_{1,2}$ is simply $(m, n) \mapsto (m, n, 1)$. The first diagram is cartesian because induces the map $(g, h, x) \mapsto (gh, h, x)$ from $G \times G \times X$ to itself which is an isomorphism. The fact that the second diagram is cartesian is obvious.

If we take a $B$-basis for $B \otimes A$ and the induced $B \otimes A$-basis for $B \otimes A \otimes A$, the first diagram shows that if $M$ is the matrix representing multiplication by $\rho(b)$, $j(M)$ is the matrix representing $\mathrm{id} \otimes m(\rho(b))$, and hence $j(\chi(t))$ is the characteristic polynomial of $\mathrm{id} \otimes m(\rho(b))$. For the same reason, the second diagram shows that $\rho(\chi(t))$ is the characteristic polynomial of $\rho \otimes \mathrm{id}(\rho(b))$. But the action defined by $\rho$ is associative, hence $\rho \otimes \mathrm{id}(\rho(b)) = \mathrm{id} \otimes m(\rho(b))$ and so $j(\chi(t)) = \rho(\chi(t))$. This means that the coefficients of $\chi(t)$ are in $B^G = C$.

Thanks to Cayley-Hamilton,

$$\rho(b)^n + j(c_{n-1})\rho(b)^{n-1} + \cdots + j(c_1)\rho(b) + j(c_0) = 0$$

and, since $j(\chi(t)) = \rho(\chi(t))$,

$$\rho(\chi(b)) = \rho(b)^n + \rho(c_{n-1})\rho(b)^{n-1} + \cdots + \rho(c_1)\rho(b) + \rho(c_0) = 0.$$

But $\rho$ is injective thanks to Corollary 2.32, and so $\chi(b) = 0$.

*Step 3*: the natural map $\varphi : (X/G)_{\mathrm{rs}} \to Y = \operatorname{Spec} C$ is an isomorphism of ringed spaces.

Define a map $N : B \to C$ by

$$N(b) = \operatorname{Norm}(\rho(b))$$

where $\rho(b) \in B \otimes A$ and $B \otimes A$ is considered as a free $B$-module. Since $c_0 = (-1)^n \operatorname{Norm}(\rho(b)) \in C$ where $\chi(t) = t^n + \cdots + c_1 t + c_0$ is the characteristic polynomial of multiplication by $\rho(b)$, we get that $N(b) \in C$. Moreover, the relation $\chi(b) = 0$ implies

$$N(b) = (-1)^{n+1} \cdot b \cdot (b^{n-1} + c_{n-1}b^{n-2} + \cdots + c_1)$$

and hence $N(b) \in \mathfrak{a} \cap C$ if $b$ is contained in an ideal $\mathfrak{a}$ of $B$.

Since $B$ is integral over $C$, $X \to Y$ is surjective, and hence $(X/G)_{\mathrm{rs}} \to Y$ is surjective, too. Let us prove that it is injective. Let $\mathfrak{p}, \mathfrak{p}' \subseteq B$ be primes such that $\mathfrak{p} \cap C = \mathfrak{p}' \cap C$, we claim that they are contained in the same orbit. Since $A$ is finite over $k$, $j : B \to B \otimes A$ is finite, and hence there is a finite number of primes $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_r \subseteq B \otimes A$ such that $j^{-1}(\mathfrak{Q}_i) = \mathfrak{p}'$. Call $\mathfrak{q}_i = \rho^{-1}(\mathfrak{Q}_i) \subseteq B$, we need to prove that $\mathfrak{p} = \mathfrak{q}_i$ for some $i$. Since $\mathfrak{q}_i \cap C = \mathfrak{p}' \cap C = \mathfrak{p} \cap C$, it is enough to show $\mathfrak{p} \subseteq \mathfrak{q}_i$ for some $i$, because $B$ is integral over $C$.

If this is not true, there exists $b \in \mathfrak{p}$ not contained in $\mathfrak{q}_1 \cup \ldots \mathfrak{q}_r$ [AM69, Proposition 1.11]. Lemma 3.43 implies that the primes of $B$ containing $N(b)$ are of the form $j^{-1}(\mathfrak{a})$, with $\mathfrak{a} \subseteq B \otimes A$ a prime containing $\rho(b)$. Since $b \in \mathfrak{p}$, $N(b) \in \mathfrak{p} \cap C = \mathfrak{p}' \cap C$, hence there is an $i$ such that $\mathfrak{Q}_i$ contains $\rho(b)$, i.e. $\mathfrak{q}_i$ contains $b$, absurd.

We have thus proved that the continuous map $\varphi : (X/G)_{\mathrm{rs}} \to Y$ is bijective. Moreover, since $X \to Y$ is closed (because $C \subseteq B$ is integral) and $X \to (X/G)_{\mathrm{rs}}$ is surjective, $\varphi$ is closed, too. This implies that $\varphi$ is an homeomorphism, and the fact that $\varphi$ identifies the structure sheaves is obvious from the definitions.

Now we shall prove point (ii). Let $\psi : B \otimes_C B \to B \otimes A$ the morphism defined by

$$\psi(b_1 \otimes b_2) = \rho(b_1)j(b_2) = \rho(b_1)(1 \otimes b_2).$$

The fact that the action is free implies that $G \times X \to X \times_Y X$ is a closed embedding, i.e. that $\psi$ is surjective.

Let $\mathfrak{q}$ be a prime ideal of $C$, we want to prove that $B_\mathfrak{q} = B \otimes_C C_\mathfrak{q}$ is a free $C_\mathfrak{q}$ module of rank $n = \dim_k A$. If we prove this for every prime $\mathfrak{q} \subseteq C$, we get that $B$ is locally free of rank $n$ over $C$. Moreover $\varphi_\mathfrak{q}$, the localization of $\varphi$ at $\mathfrak{q}$ as a map of $C$-modules, is a surjective map between free modules of the same rank and hence is an isomorphism, and this implies that also $\varphi$ is an isomorphism. Call $\mathfrak{r} \subseteq B_\mathfrak{q}$ the Jacobson radical, $B_\mathfrak{q}/\mathfrak{r}$ is a finite product of fields because $B_\mathfrak{q}$ is finite over $A_\mathfrak{q}$.

*Case 1*: $k(\mathfrak{q}) = C_\mathfrak{q}/\mathfrak{q}C_\mathfrak{q}$ is infinite.

Consider the $C_\mathfrak{q}$-submodule

$$N := \{\rho(b) | b \in B_\mathfrak{q}\} \subseteq M := B_\mathfrak{q} \otimes A.$$

Since $\varphi_\mathfrak{q} : B_\mathfrak{q} \otimes_{C_\mathfrak{q}} B_\mathfrak{q} \to B_\mathfrak{q} \otimes A$ is surjective, $N$ spans $M$ as a $B_\mathfrak{q}$-module. We have that $B_\mathfrak{q}/\mathfrak{r}$ is a finite product of fields containing $k(p)$, let us prove that $N/\mathfrak{r}N$ contains a basis of $M/\mathfrak{r}M$ as a free $B_\mathfrak{r}/\mathfrak{r}$-module.

Let $n_1, \ldots, n_r \in N/\mathfrak{r}N$ generate $M/\mathfrak{r}M$ as a $B_\mathfrak{r}/\mathfrak{r}$-module. Fix a matrix $\lambda = (\lambda_{ij}) \in k(p)^{rn}$ with $i = 1, \ldots, r$, $j = 1, \ldots, n$. Consider the element

$$m_{j,\lambda} = \lambda_{1j} n_1 + \cdots + \lambda_{rj} n_r \in N/\mathfrak{r}N$$

Now, let $B_\mathfrak{q}/\mathfrak{r} = F_1 \times \cdots \times F_s$ with $F_l$ field. Let $1_l \in F_l$ be the identity, we have that $m = 1_1 m + \cdots + 1_s m$ for every $m \in M/\mathfrak{r}M$. Since $1_l n_1, \ldots, 1_l n_r$ generate $F_l N/\mathfrak{r}N$ as an $F$-module, there is an open nonempty subset $U_l \subseteq k(p)^{rn}$ such that $1_l m_{1,\lambda}, \ldots, 1_l m_{j,\lambda}$ for every $\lambda \in U_l$. Since $k(p)$ is infinite, $k(p)^{rn}$ is irreducible, and hence there exist $\lambda_0 \in U_1 \cap \cdots \cap U_s$. But then, $m_{1,\lambda_0}, \ldots, m_{j,\lambda_0}$ is a basis for $M/\mathfrak{r}M$ as an $B_\mathfrak{q}/\mathfrak{r}$-module.

Hence, we have found that $N/\mathfrak{r}N$ contains a basis of $M/\mathfrak{r}M$ as a free $B_\mathfrak{q}/\mathfrak{r}$-module. This implies that, thanks to the Nakayama lemma, $N$ contains a basis of $M$ as a $B_\mathfrak{q}$-module, i.e. that there exist $b_1, \ldots, b_n \in B_\mathfrak{q}$ such that $\rho(b_1), \ldots, \rho(b_n)$ form a basis of $B_\mathfrak{q} \otimes A$: we claim that $b_1, \ldots, b_n$ are a basis of $B_\mathfrak{q}$ as a $C_\mathfrak{q}$-module. For every $b \in B_\mathfrak{q}$, we have unique $x_1, \ldots, x_n \in B_\mathfrak{q}$ such that

$$\rho(b) = x_1 \rho(b_1) + \cdots + x_n \rho(b_n) =$$

$$= x_1 \otimes 1 \cdot \rho(b_1) + \cdots + x_n \cdot \otimes 1 \rho(b_n).$$

Since $\rho$ is injective thanks to Corollary 2.32, it is enough to show that $x_i \otimes 1 \in \rho(C_\mathfrak{q})$.

In order to do this, consider $B_\mathfrak{q} \otimes A \otimes A$ as a module over $B_\mathfrak{q} \otimes A$ via the homomorphism $j_{1,2}$ given by $b \otimes a \mapsto b \otimes a \otimes 1$. The cartesian diagram

$$\begin{tikzcd} B_{\mathfrak{q}} \otimes A \arrow[r, "\mathrm{id} \otimes m"] & B_{\mathfrak{q}} \otimes A \otimes A \\ B_{\mathfrak{q}} \arrow[u, "j"] \arrow[r, "j"] & B \otimes A \arrow[u, "j_{1,2}"'] \end{tikzcd}$$

shows that

$$\gamma_i = \mathrm{id} \otimes m(\rho(b_i)) = \rho \otimes \mathrm{id}(\rho(b_i))$$

is a basis of $B_{\mathfrak{q}} \otimes A \otimes A$ over $B_{\mathfrak{q}} \otimes A$. Moreover,

$$\begin{tikzcd} \mathrm{id} \otimes m(\rho(b)) \arrow[r, equal] \arrow[d, equal] & (x_1 \otimes 1 \otimes 1)\,\gamma_1 + \cdots + (x_n \otimes 1 \otimes 1)\,\gamma_n \\ \rho \otimes \mathrm{id}(\rho(b)) \arrow[r, equal] & \big(\rho(x_1) \otimes 1\big)\,\gamma_1 + \cdots + \big(\rho(x_n) \otimes 1\big)\,\gamma_n \end{tikzcd}$$

and hence $j(x_i) = x_i \otimes 1 = \rho(x_i)$, and $x_i \in C_q$, as desired.

*Case 2*: $k(p)$ is finite.

If we find a local ring $(C', \mathfrak{m})$ such that $C'/\mathfrak{m}$ is infinite and with a faithfully flat homomorphism $C_{\mathfrak{q}} \to C'$, we may reduce to case 1 to show that $B_{\mathfrak{q}} \otimes_{C_{\mathfrak{q}}} C'$ is free of rank $n$ over $C'$ and then apply Proposition 3.7 to conclude. In fact, call $B' = B_{\mathfrak{q}} \otimes_{C_{\mathfrak{q}}} C'$, $\rho$ induces an $A$-comodule structure $\rho'$ on $B'$ such that $C' \subseteq B'$ is $B'^G$. In fact, we have an exact sequence of $C_{\mathfrak{q}}$ modules

$$0 \to C_{\mathfrak{q}} \to B_{\mathfrak{q}} \xrightarrow{\rho - j} B_{\mathfrak{q}} \otimes A$$

and, tensoring with $C'$, we get an exact sequence of $C'$ modules

$$0 \to C' \to B' \xrightarrow{\rho' - j'} B' \otimes A.$$

As $C'$, we may take the strict henselianization of $C_{\mathfrak{q}}$ ([Ray70, Théorème VIII.2.2] and [Ray70, Théorème VIII.4.3]). $\square$

**Corollary 3.46.** *Let $G = \operatorname{Spec} A$ be a finite, closed subgroup of an affine group-scheme $H = \operatorname{Spec} B$, acting on $H$ by multiplication on the right. The geometric quotient $q : H \to H/G$ exists and is flat, and there exists a rational point $p$ such that $G$ is $q^{-1}(p)$. We write $[G]$ for $p$.*

*Proof.* Since the action of $G$ on $H$ is clearly free, thanks to Theorem 3.45 the geometric quotient $q : H \to H/G = \operatorname{Spec} C$ exists and is affine. Let $p \in H/G(k)$ be the image of the identity $\varepsilon : \operatorname{Spec} k \to H$. Since the image of $C \to B \to A$ is $G$-invariant and $A$ is the Hopf algebra of $G$, we have that $C \to A$ splits as $C \to k = A^G \to A$. This means that $G \to H/G$ splits as $G \to \operatorname{Spec} k \xrightarrow{p} H/G$, and hence we have a map $G \to q^{-1}(p)$ that is a closed embedding because $G \to q^{-1}(p) \to H$ is a closed embedding.

$$\begin{tikzcd}
G \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& q^{-1}(p) \arrow[r] \arrow[d] & p \arrow[d] \\
& H \arrow[r] & H/G
\end{tikzcd}$$

But, thanks to point (ii) of Theorem 3.45, $\dim_k \mathrm{H}^0(q^{-1}(p)) = \dim_k \mathrm{H}^0(A)$ and hence $G = q^{-1}(p)$ because everything is affine. $\square$

## 3.5 Torsors and étale coverings

In this section, we are going to make a comparison between torsors and étale coverings, in order to compare Grothendieck's and Nori's versions of the fundamental group in the next chapter.

**Definition 3.47.** A morphism $\pi : E \to X$ is an étale covering if it is both finite and étale.

For the rest of the section, assume that $X$ is connected and has a geometric point $x_0 \in X(\Omega)$, where $\Omega$ is an algebraically closed field containing $k$. Since $\pi : E \to X$ is finite étale, $E_{x_0} \to \operatorname{Spec} \Omega$ is finite étale too, and hence it is simply a disjoint union of a finite number of copies of $\operatorname{Spec} \Omega$. If $E$ is connected and the group $\operatorname{Aut}(E/X)$ acts transitively on $E_{x_0}$, we will say that the covering is Galois.

**Proposition 3.48.** *A $G$-torsor $T \to X$ is an étale covering if and only if $G$ is finite étale.*

*Proof.* Take $U \to T$ a faithfully flat and quasi compact morphism such that $T|_U$ is trivial, for example $T = U$. Then, $G \times U \to U$ is finite étale if and only if $G$ is finite étale, and hence the thesis descends from Proposition 3.7. $\square$

**Proposition 3.49.** *1. If $G$ is a finite discrete group-scheme, connected $G$-torsors over $X$ coincide with étale Galois coverings with automorphism group $G$.*

*2. If $k$ has characteristic 0, a finite torsor $T \to X$ is an étale covering. Moreover, if $T$ is geometrically connected, there exists a finite separable extension $L/k$ such that $T_L \to X_L$ is a Galois covering.*

*Proof.* 1. Consider an étale Galois covering $\pi : E \to X$ and the discrete group-scheme $G$ associated to $\mathrm{Aut}(E/X)$. As a scheme, it is simply the disjoint union of $n$ copies of $\operatorname{Spec} k$, where $n$ is the cardinality of $\mathrm{Aut}(E/X)$. There is a natural action

$$\alpha : G \times E = \bigsqcup_{\sigma \in \mathrm{Aut}(E/X)} E \to E$$

defined by $\sigma_i$ on the copy of $E$ associated to $\sigma_i$.

The projection $\pi : E \to X$ is $G$-invariant, affine and flat. Since $\pi$ is finite and flat, thanks to [EGAIV-2, Theorem 2.4.6] it is open, and since $X$ is connected, it is surjective. We only need to prove that $\delta_\alpha : G \times E \to E \times_X E$ is an isomorphism.

Consider now the diagonal $\Delta = \mathrm{id} \times \mathrm{id} : E \to E \times_X E$. The projection $\pi$ is affine and hence separated, and so $\Delta$ is a closed immersion. Thanks to [Mur67, Proposition 3.3.2], $\Delta$ is also an open immersion, hence we have embedded $E$ as an open and closed subscheme of $E \times_X E$. If we take $\mathrm{id} \times \sigma : E \to E \times_X E$ with $\sigma \in \mathrm{Aut}(E/X)$ instead of $\Delta$, the same is true, because $\mathrm{id} \times \sigma$ is the composition of $\Delta$ with an automorphism of $E \times_X E$. Fix a point $c_0 \in E_{x_0}$. These copies of $E$ embedded in $E \times_X E$ are different, because the point $(c_0, \sigma(c_0))$ is in the open subscheme $(\mathrm{id} \times \sigma')(E)$ if and only if $\sigma = \sigma'$: if $(c_0, \sigma(c_0)) \in (\mathrm{id} \times \sigma')(E)$, $\sigma(c_0) = \sigma'(c_0)$, and this implies $\sigma = \sigma'$ thanks to [Mur67, Lemma 4.4.1.6(iii)].

It is clear now that $\delta_\alpha$ identifies $G \times E$ with an open and closed subscheme of $E \times_X E$. Hence, call $E' = (E \times_X E) \setminus (G \times E)$, which is an open and closed subscheme, too. Our claim is that $E'$ is empty.

Composition and base change of finite étale morphisms are finite étale, hence $E \times_X E \to X$ is finite étale. Moreover, closed immersion are finite, and open immersion are étale, hence $E' \to X$ is finite étale, too. If $E'$ is nonempty, its fiber $E'_{x_0} \subseteq (E \times_X E)_{x_0}$ is nonempty, because $X$ is connected and hence étale coverings are surjective. But since $E$ is Galois, every element of $(E \times_X E)_{x_0}$ is of the form $(\sigma(c_0), \sigma'(c_0))$ and is contained in the open subscheme $\mathrm{id} \times (\sigma' \circ \sigma^{-1})(S)$ for some $\sigma, \sigma' \in \mathrm{Aut}(E/X)$.

On the other hand, let $T \to X$ be a torsor over a finite discrete group-scheme $G$. Then $G$ is finite étale over $k$, and hence $T \times_X T \simeq G \times T \to T$ is finite étale. Using flat descent (Proposition 3.7), this implies that $T \to X$ is finite étale, too. Since $G$ is discrete, the fact that the covering is Galois is immediate: the group over which is defined $G$ acts by automorphisms on $T$, and consequently on the fiber $T_{x_0} \simeq G_\Omega$ in the obvious way, which is transitive.

2. Now, let $G$ be a finite group-scheme and $T \to X$ a $G$-torsor. As shown in [Wat79, sect. 11.4], if $k$ has characteristic 0, $G$ is finite étale over $k$, and we may conclude as above that $T \to X$ is finite étale. Since $G$ is finite étale over $k$, there exists $L$ such that $G_L$ is discrete, and hence $T_L$ is Galois thanks to point 1.

□

# Chapter 4

# The fundamental group-scheme

In this chapter we want to find when a scheme has a fundamental group-scheme, i.e. a profinite group-scheme whose finite quotients classify torsors over the scheme.

Let us fix a scheme $X$ over a base field $k$ with a $k$-rational point $x_0$. Now consider the category $\mathcal{FT}(X)_{x_0}$ whose objects are triples $(T, G, t_0)$ where $T \to X$ is a $G$-torsor, $G$ a finite group-scheme and $t_0$ a $k$-rational point of $T$ over $x_0$. A morphism $(f, g) : (T, G, t_0) \to (T', G', t'_0)$ is a morphism of torsors sending $t_0$ to $t'_0$

By $\mathcal{PT}(X)_{x_0}$ we will denote the category of triples as above, except that now we allow $G$ to be a profinite group-scheme.

**Definition 4.1.** A profinite group $\pi_1^N(X, x_0)$ is a *fundamental group-scheme* of $X$ if there exists a triple $(\widetilde{T}, \pi_1^N(X, x_0), \widetilde{t}_0)$ in $\mathcal{PT}(X)_{x_0}$ such that for every object $(T, G, t_0)$ there is a unique morphism $(\widetilde{T}, \pi_1^N(X, x_0), \widetilde{t}_0) \to (T, G, t_0)$. We call $\widetilde{T}$ the *universal torsor* of $X$.

**Lemma 4.2.** *To check that $(\widetilde{T}, \pi_1^N(X, x_0), \widetilde{t}_0)$ is an initial object of $\mathcal{PT}(X)_{x_0}$, it is enough to verify the existence of a unique morphism $(\widetilde{T}, \pi_1^N(X, x_0), \widetilde{t}_0) \to (T, G, t_0)$ when $G$ is finite.*

*Proof.* Let $(T, G, t_0)$ be an object in $\mathcal{PT}(X)_{x_0}$, where $G = \varprojlim_{i \in \mathcal{P}} G_i$ is a profinite group-scheme, with $\mathcal{P}$ a cofiltered category. Thanks to Corollary 3.23, the torsors $T_i$ induced by $G \to G_i$ form a projective system $\mathcal{P} \to \mathcal{PT}(X)_{x_0}$ with limit $(T, G, t_0)$. Let us suppose that there exists a unique morphism $(\widetilde{T}, \pi_1^N(X, x_0), \widetilde{t}_0) \to (T_i, G_i, t_{0,i})$ for every $i$, where $t_{0,i}$ is the image of $t_0$ in $T_i$. Then, uniqueness implies that these morphisms define a cone for $\mathcal{P} \to \mathcal{PT}(X)_{x_0}$ inducing a unique morphism $(\widetilde{T}, \pi_1^N(X, x_0), \widetilde{t}_0) \to (T, G, t_0)$. □

## 4.1 Existence of the fundamental group-scheme

### 4.1.1 Fibered product of torsors

We claim that $X$ has a fundamental group-scheme exactly when $\mathcal{FT}(X)_{x_0}$ is closed under finite products, i.e. when

$$(T_1 \times_T T_2, G_1 \times_G G_2, t_{0,1} \times t_{0,2}) = (T^\times, G^\times, t_0^\times)$$

is an object of $\mathcal{FT}(X)_{x_0}$ for every pair of morphisms

$$(f_i, \rho_i)(T_i, G_i, t_{0,i}) \to (T, G, t_0)$$

with $i = 1, 2$.

Before starting, we need a technical lemma.

**Lemma 4.3.** *A finite morphism $Y \to X$ is a closed embedding if and only if the diagonal $\Delta : Y \to Y \times_X Y$ is an isomorphism.*

*Proof.* It is obvious that $\Delta$ is an isomorphism if $Y \to X$ is a closed embedding.

Now, suppose that $\Delta$ is an isomorphism. Finite morphisms are affine and the problem is local, hence we may suppose $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. We have a finite homomorphism $A \to B$, we know that $B \otimes_A B \to B$ is an isomorphism and we want to show that the image of $A \to B$ is $B$. Call $I$ the kernel of $A \to B$, $B \otimes_A B \simeq B \otimes_{A/I} B$, hence we may replace $A$ with $A/I$ and suppose $A \subseteq B$.

*Case 1*: $A$ is a field. If $B \otimes_A B \simeq B$, $\dim_A B = 1$, hence $A = B$.

*Case 2*: $A$ is local. Let $\mathfrak{m} \subseteq A$ be the maximal ideal. The following diagram is cartesian:

$$\begin{array}{ccc} A & \longrightarrow & B \otimes_A B \\ \downarrow & & \downarrow \\ A/\mathfrak{m} & \longrightarrow & B/\mathfrak{m}B \otimes_{A/\mathfrak{m}} B/\mathfrak{m}B \end{array}$$

Since $\Delta \otimes \mathrm{id} : (B \otimes_A B) \otimes A/\mathfrak{m} \to B \otimes_A A/\mathfrak{m}$ is an isomorphism and the diagram above is cartesian,

$$\Delta_{A/\mathfrak{m}} : B/\mathfrak{m}B \otimes_{A/\mathfrak{m}} B/\mathfrak{m}B \to B/\mathfrak{m}B$$

is an isomorphism, too. Thanks to case 1, $A/\mathfrak{m} = B/\mathfrak{m}B$ and hence $A + \mathfrak{m}B = B$, and this implies $A = B$ thanks to Nakayama's lemma [AM69, Corollary 2.7].

*Case 3*: $A$ is a commutative ring. To show that $A = B$, it is enough to show that $A_\mathfrak{p} = B_\mathfrak{p}$ for every prime $\mathfrak{p} \subseteq A$, and this is case 2. In fact, $B_\mathfrak{p} \simeq B \otimes_A A_\mathfrak{p}$ and hence

$$B_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{p} = (B \otimes_A B) \otimes_A A_\mathfrak{p} \to B \otimes_A A_\mathfrak{p} = B_\mathfrak{p}$$

is an isomorphism. □

**Proposition 4.4.** *With notations as above, $T^\times$ is a torsor over a closed subscheme $Y$ of $X$ containing $x_0$.*

*Proof.* We will divide the proof in three steps.

*Step 1*: $G^\times \times T^\times \simeq T^\times \times_X T^\times$.

Call $\overline{T} = T_1 \times_X T_2$, $\overline{T}$ is a $\overline{G} = G_1 \times G_2$ torsor and we have an obvious map $T^\times \to \overline{T}$ equivariant with respect to $G^\times \to \overline{G}$. Let $\mathrm{p}_i$ be the composition $\overline{T} \to T_i \to T$. The Yoneda Lemma tells us that there exists a unique morphism $z : \overline{T} \to G$ such that $\mathrm{p}_1 = z \cdot \mathrm{p}_2$. If $\varepsilon : \operatorname{Spec} k \to G$ is the identity, we have that $T^\times \to \overline{T}$ is the closed subscheme $z^{-1}(\varepsilon) \subseteq \overline{T}$: in fact, given a scheme $U$ and a point $t \in \overline{T}(U)$, $\mathrm{p}_1(t) = \mathrm{p}_2(t)$ if and only if $z(t) = \varepsilon$. The isomorphism $\overline{G} \times \overline{T} \xrightarrow{\sim} \overline{T} \times_X \overline{T}$ identifies the respective closed subschemes $G^\times \times T^\times$ and $T^\times \times_X T^\times$.

In fact, using the Yoneda Lemma, a point $(g_1, g_2, t_1, t_2)$ of $\overline{G} \times \overline{T}$ is in $G^\times \times T^\times$ if and only if $\rho_1(g_1) = \rho_2(g_2)$ and $f_1(t_1) = f_2(t_2)$, and $(t_1', t_2', t_1'', t_2'') \in \overline{T} \times_X \overline{T}$ is in $T^\times \times_X T^\times$ if and only if $f_1(t_1') = f_2(t_2')$, $f_1(t_1'') = f_2(t_2'')$. Then, our claim descends directly from the fact that

$$(g_1, g_2, t_1, t_2) \mapsto (t_1, t_2, g_1 t_1, g_2 t_2).$$

*Step 2*: there exists a scheme $Y$ and a $G^\times$-invariant morphism $T^\times \to Y$ making $T^\times$ a $G^\times$-torsor.

Theorem 3.45 gives us a faithfully flat, affine, geometric quotient $T^\times \to Y$ such that $G^\times \times T^\times \simeq T^\times \times_Y T^\times$. To apply the theorem, we need to check two hypotheses:

- the orbit of any point is contained in an affine open subset of $T^\times$,
- the action is free, i.e. $G^\times \times T^\times \to T^\times \times T^\times$, $(g, t) \mapsto (t, gt)$ is a closed embedding.

The first one is true because $T^\times \to X$ is $G^\times$-invariant and affine: if $U \subseteq X$ is affine, its inverse image in $T^\times$ is open, affine and $G^\times$-invariant, and we may cover $T^\times$ with such sets. The second one is true because $T^\times \times_X T^\times \to T^\times \times T^\times$ is a closed immersion and $G^\times \times T^\times \to T^\times \times_X T^\times$ is an isomorphism. Since $T^\times \to X$ is invariant and $T^\times \to Y$ is a categorical quotient, we obtain a morphism $Y \to X$.

*Step 3*: $Y \to X$ is a closed embedding.

As $\mathcal{O}_X$-algebras, $\mathcal{O}_{T^\times}$ is finite over $\mathcal{O}_X$ and $\mathcal{O}_Y$ is contained in $O_{T^\times}$: this implies that $Y$ is finite over $X$, too. Thanks to Lemma 4.3, it is enough to check that $\Delta : Y \to Y \times_X Y$ is an isomorphism. In order to do this, consider the following commutative diagram:

$$\begin{tikzcd} G^\times \times T^\times \arrow[r, "\sim"] \arrow[d] & T^\times \times_X T^\times \arrow[d] \\ Y \arrow[r, "\Delta"] & Y \times_X Y \end{tikzcd}$$

where the first row is the isomorphism $(g, t) \mapsto (t, gt)$. The second column $T^\times \times_X T^\times \to Y \times_X Y$ is a torsor with respect to the obvious action of $G^\times \times G^\times$. On $G^\times \times T^\times$, consider the action of $G^\times \times G^\times$

$$(g_1, g_2) \times (g, t) \mapsto (g_2 g g_1^{-1}, g_1 t).$$

This action makes $G^\times \times T^\times$ a torsor over $Y$: $G^\times \times T^\times \to Y$ is faithfully flat and affine because $T^\times \to Y$ is faithfully flat and affine, and

$$(G^\times \times G^\times) \times (G^\times \times T^\times) \simeq (G^\times \times T^\times) \times_Y (G^\times \times T^\times)$$

thanks to the Yoneda Lemma. Moreover, the isomorphism $G^\times \times T^\times \to T^\times \times_X T^\times$ is $G^\times \times G^\times$-equivariant, and hence $Y \to Y \times_X Y$ is an isomorphism, too: geometric quotients are unique, and torsors are geometric quotients thanks to Lemma 3.39.

Finally, $x_0 \in Y$ because $t_{0,1} \times t_{0,2}$ is a point of $T^\times$ over $x_0$. □

**Proposition 4.5.** *$X$ has a fundamental group-scheme if and only if $\mathcal{FT}(X)_{x_0}$ is closed under finite products.*

*Proof.* Let us suppose that $(\widetilde{T}, \pi_1^N(X, x_0), \widetilde{t}_0)$ is an initial object of $\mathcal{PT}(X)_{x_0}$. Consider a pair of morphism

$$(f_i, \rho_i)(T_i, G_i, t_{0,i}) \to (T, G, t_0)$$

in $\mathcal{FT}(X)_{x_0}$, with $i = 1, 2$, and let $T^{\times} = T_1 \times_T T_2$ be a torsor over a closed subscheme $Y \to X$. By definition, there exist morphisms

$$(r_i, s_i) : (\widetilde{T}, \pi_1^N(X, x_0), \widetilde{t}_0) \to (T_i, G_i, t_{0,i})$$

for $i = 1, 2$, and by uniqueness

$$(f_1 r_1, \rho_1 s_1) = (f_2 r_2, \rho_2 s_2) : (\widetilde{T}, \pi_1^N(X, x_0), \widetilde{t}_0) \to (T, G, t_0).$$

Hence we have a morphism $\widetilde{T} \to T^{\times}$, and this implies $Y = X$: the composition of morphisms of $\mathcal{O}_X$ algebras

$$\mathcal{O}_X \to \mathcal{O}_Y \to \mathcal{O}_{T^{\times}} \to \mathcal{O}_{\widetilde{T}}$$

is injective, and hence $\mathcal{O}_X \to \mathcal{O}_Y$ is injective, too.

On the other hand, let us suppose that $\mathcal{FT}(X)_{x_0}$ is closed under finite products. We have that $\mathcal{FT}(X)_{x_0}$ is cofiltered:

- $\mathcal{FT}(X)_{x_0}$ is nonempty, $X \to X$ is a torsor.
- If $T_1$, $T_2$ are finite torsors, $T_1 \times_X T_2$ is a finite torsor, too, and we have morphisms $T_1 \times_X T_2 \to T_1$, $T_1 \times_X T_2 \to T_2$.
- If we have two morphisms of torsors $f, g : T' \to T$, then the two compositions $f \circ \mathrm{p}_1, g \circ \mathrm{p}_2 : T' \times_T T' \to T' \to T$ are equal.

Now, let $\mathcal{FT}(X)_{x_0}^{\mathrm{op}} \to \mathrm{QCoh}(X)$ be the direct system $(T, G, t_0) \mapsto \mathcal{O}_T$, thanks Proposition 2.52 it defines a colimit quasi-coherent sheaf $\mathcal{A}$ which inherits the structure of $\mathcal{O}_X$-algebra. Call $\widetilde{T}$ the relative spectrum $\operatorname{Spec} \mathcal{A}$, $\widetilde{t}_0$ the rational point induced by the cone $(T, G, t_0) \mapsto (t_0 : \operatorname{Spec} k \to T)$ and $\pi_1^N(X, x_0)$ the projective limit of the forgetful functor $\mathcal{FT}(X)_{x_0}^{\mathrm{op}} \to \mathrm{AffGrp}_k$. We have an induced action $\pi_1^N(X, x_0) \times \widetilde{T} \to \widetilde{T}$ and a $\pi_1^N(X, x_0)$-invariant morphism $\widetilde{T} \to X$ such that $\pi_1^N(X, x_0) \times \widetilde{T} \to \widetilde{T} \times_X \widetilde{T}$ is an isomorphism: limits commute with products, and so the action $G \times T \to T$ and the isomorphisms $G \times T \to T \times_X T$ pass to the limit.

Let us show that $\widetilde{T} \to X$ is surjective. For every $(T, G, t_0) \in \mathcal{FT}(X)_{x_0}$, $\mathcal{O}_X \to \mathcal{O}_T$ is injective. The construction of the limit $\mathcal{O}_{\widetilde{T}}$ contained in Proposition 2.52.vii shows that $\mathcal{O}_X \to \mathcal{O}_{\widetilde{T}}$ is injective, too: take an affine open subset $U \subseteq X$, $\mathcal{O}_{\widetilde{T}}(U) \simeq \operatorname{colim}_T \mathcal{O}_T(U)$ because $U$ is quasi-compact, hence a section $f \in \mathcal{O}_X(U)$ has image 0 in $\mathcal{O}_{\widetilde{T}}(U)$ if and only if there exists a torsor $T$ such that the image of $f$ in $\mathcal{O}_T(U)$ is 0.

Moreover, if we take a section $s \in \mathcal{O}_{\widetilde{T}}(U)$, there exists a torsor $T$ such that $s$ is in the image of $\mathcal{O}_T(U) \mapsto \mathcal{O}_{\widetilde{T}}(U)$. But $\mathcal{O}_T(U)$ is finite over $\mathcal{O}_X(U)$, hence $\mathcal{O}_{\widetilde{T}}$ is integral over $\mathcal{O}_X$, and $\widetilde{T} \to X$ is surjective.

Finally, Lemma 4.6 implies that $\widetilde{T} \to X$ is flat. $\square$

**Lemma 4.6.** *Let $\mathcal{D} \to \mathrm{QCoh}(X)$ be a direct system $i \mapsto S_i$ of quasi-coherent sheaves with limit $S$. If $S_i$ is flat for every $i$, then $S$ is flat.*

*Proof.* The problem is local, we may suppose $X = \operatorname{Spec} R$, $S_i = \widetilde{M}_i$, $S = \widetilde{M}$ with $M = \operatorname{colim}_i M_i$. Let $N \to N'$ be an injective map of $R$-modules, we need to show that $M \otimes N \to M \otimes N'$ is injective.

Call $\mathcal{K}$ the category with three objects $A, B, C$ and only two morphisms $A \to B$ and $C \to B$, excluding the identities. We can think to the kernel of $M \otimes N \to M \otimes N'$ as the limit of a diagram $\mathcal{K} \to \mathrm{Mod}_R$ sending $A \mapsto N$, $B \mapsto N'$, $C \mapsto 0$. We have an obvious embedding $\mathrm{Mod}_R \to \mathrm{Set}$ respecting direct limits (as can be seen in the proof of Proposition 2.52.vi) and kernels defined using $\mathcal{K}$ (obvious). Hence, thanks to Proposition 2.53,

$$\ker(M \otimes N \to M \otimes N') = \operatorname*{colim}_i \ker(M_i \otimes N \to M_i \otimes N') = 0.$$

□

### 4.1.2 Reduced and connected base

Now we are going to show that if $X$ is reduced and connected then $\mathcal{FT}(X)_{x_0}$ is closed under finite products.

**Lemma 4.7.** *Let $T \to X$ be a $G$-torsor. If $G = \operatorname{Spec} A$ is of finite type over $k$, then $T \to X$ is locally of finite presentation.*

*Proof.* Let $U \to T$ a faithfully flat and quasi-compact morphism trivializing $T$, for example $U = T$. Thanks to Proposition 3.7, it is enough to show that $G \times U \to U$ is locally of finite presentation. But this is immediate, because $G \to \operatorname{Spec} k$ is locally of finite presentation since $A$ is of finite type over $k$. □

**Theorem 4.8.** *If $X$ is connected and reduced, it has a fundamental group-scheme.*

*Proof.* We need to show that $\mathcal{FT}(X)_{x_0}$ is closed under finite products. With notation as above, consider two morphisms

$$(f_i, \rho_i)(T_i, G_i, t_{0,i}) \to (T, G, t_0)$$

with $i = 1, 2$. We know that $T^{\times} = T_1 \times_T T_2$ is a torsor over a closed subscheme $Y \to X$ and we have a morphism $z : \overline{T} = T_1 \times_X T_2 \to G$ such that $T^{\times} = z^{-1}(\varepsilon)$, with $\varepsilon : \operatorname{Spec} k \to G$ the identity.

Since $G$ is finite, the connected component of the identity $G^{\circ}$ is open and closed. In fact, consider $G = \operatorname{Spec} A$ and $\pi_0(A) = k_0 \times \cdots \times k_n \subseteq$

$A$, with $k_i/k$ separable extensions and the projection $\pi_0(A) \to k_0 = k$ corresponding to the identity $\varepsilon : \operatorname{Spec} k \to \pi_0(G)$. Then, $G^\circ = \operatorname{Spec} k_0 A = \operatorname{Spec} A_{(1,0,\ldots,0)}$.

Since $G^\circ$ is open and closed, $z^{-1}(G^\circ)$ is open and closed, too. We know $\pi : \overline{T} \to X$ is finite and flat and locally of finite presentation because $G$ is finite, hence $\pi(z^{-1}(G^\circ))$ (as a set) is open and closed thanks to [EGAIV-2, Theorem 2.4.6]. This implies $Y = \pi(z^{-1}(G^\circ)) = X$ because $X$ is connected and $Y$ is nonempty (it contains $x_0$). Since $G$ is finite, we also know that $\varepsilon = G^\circ$ as sets, and hence, as sets,

$$Y = \pi(T^\times) = \pi(z^{-1}(\varepsilon)) = \pi(z^{-1}(G^\circ)) = X.$$

Finally, if $Y = X$ as sets and $X$ is reduced, we have $Y = X$ as schemes. □

**Proposition 4.9.** *A morphism $f : (X, x_0) \to (Y, y_0)$ of pointed schemes with fundamental group induces a natural homomorphism of group-schemes $\pi_1^N(X, x_0) \to \pi_1^N(Y, y_0)$.*

*Proof.* If $T \to Y$ is a $G$-torsor, $T \times_Y X$ is a $G$-torsor, too. The association $T \mapsto T \times_Y X$ defines a functor $f^* : \mathcal{FT}(Y)_{y_0} \to \mathcal{FT}(X)_{x_0}$ preserving the forgetful functor on $\mathrm{AffGrp}_k$. This induces an homomorphism of affine group-schemes $f_* : \pi_1^N(X, x_0) \to \pi_1^N(Y, y_0)$. If $g : (Y, y_0) \to (Z, z_0)$ is another morphism, there is an isomorphism of functors $f^* g^* \simeq (gf)^*$, and hence $g_* f_* = (gf)_*$. □

## 4.2 Reduction of structure group

**Definition 4.10.** Let $T \to X$ be a $G$-torsor and $H \subseteq G$ a closed subgroup. A $H$-torsor $T'$ is a *reduction of structure group* of $T$ to $H$ if there exists a $H$-equivariant morphism $T' \to T$ over $X$.

**Proposition 4.11.** *Let $G$ be a group-scheme, $T \to X$ a $G$-torsor and $H \subseteq G$ a finite, closed subgroup. If there exists a $G$-equivariant morphism $f : T \to G/H$, then $T' = f^{-1}([H])$ is a reduction of structure group of $T$ to $H$.*

*Proof.* Since $[H]$ is fixed by $H$ and $T \to G/H$ is $H$-equivariant, the image of $\alpha : H \times T' \to T$ is contained in $f^{-1}([H]) = T'$, too, and hence $T'$ is $H$-invariant.

Now, we want to show that $T'$ is a $H$-torsor. The diagram

$$\begin{tikzcd}
T' \arrow[r] \arrow[d] & T \arrow[d] \\
X \arrow[r, "\mathrm{id}\times[H]"] \arrow[d] & X \times G/H \arrow[d] \\
\operatorname{Spec} k \arrow[r, "[H]"] & G/H
\end{tikzcd}$$

is cartesian, hence it is enough to show that $T \to X \times G/H$ is a $H$-torsor. Thanks to point (ii) of Theorem 3.45, this is equivalent to proving that $X \times G/H \simeq H\backslash T$.

We have a diagram

$$\begin{tikzcd}
H\backslash T \arrow[rrd, bend left] \arrow[rd, dashed] \arrow[rdd, bend right] & & \\
& X \times G/H \arrow[r] \arrow[d] & G/H \arrow[d] \\
& X \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

that gives us a morphism $H\backslash T \to X \times G/H$, we need to find an inverse. The composition

$$\psi : T \xrightarrow{(i\circ f)\times \mathrm{id}} G/H \times T \xrightarrow{\alpha} H\backslash T,$$

where $i : G/H \to G/H$ is the inverse, is $G$-invariant: if $S$ is a scheme, $p \in P(S)$ and $g \in G(S)$,

$$\psi(gp) = \alpha(i \circ f(gp), gp) = \alpha(i(gf(p)), gp) =$$
$$= \alpha(f(p)^{-1}g^{-1}, gp) = \alpha(f(p)^{-1}, p) = \psi(p).$$

Hence, $\psi$ descends to a section $\psi_0 : X \to H\backslash T$ of the projection $H\backslash T \to X$. Moreover, $f \circ \psi_0 : X \to G/H$ is constant on $g_0 \in G/H(k)$: if $p \in T(S)$,

$$f(\psi(p)) = f(f(p)^{-1} \cdot p) = f(p)^{-1} f(p).$$

This implies that the morphism $X \times G/H \to H\backslash T$ defined by $(g, x) \mapsto g\psi_0(x)$ is the inverse we where searching. □

**Definition 4.12.** A torsor $T \to X$ is *Nori-reduced* if there exists no nontrivial reduction of structure group of $T$.

**Lemma 4.13.** *Let $X$ be a scheme and $x_0 \in X(k)$ a rational point such that $\pi_1^N(X, x_0)$ exists. A finite $G$-torsor $T \to X$ is Nori-reduced if and only if the corresponding homomorphism $\pi_1^N(X, x_0) \to G$ is surjective.*

*Proof.* Let us suppose that $T$ is Nori-reduced, and call $H \subseteq G$ the image of $\pi_1^N(X, x_0) \to G$. The morphism $\pi_1^N(X, x_0) \to H$ corresponds to a $H$-torsor $T'$ with a $H$-equivariant morphism $T' \to T$: since $T$ is Nori-reduced, $T' = T$ and $H = G$.

On the other hand, let us suppose that $\pi_1^N(X, x_0) \to G$ is surjective, and take a closed subgroup $H \subseteq G$ with $T'$ a reduction of structure group of $T$ to $H$. Then, $T'$ induces an homomorphism of group-schemes $\pi_1^N(X, x_0) \to H$ making the diagram

$$\begin{tikzcd}
\pi_1^N(X, x_0) \arrow[rr] \arrow[dr] & & G \\
& H \arrow[ur] &
\end{tikzcd}$$

commute. But $\pi_1^N(X, x_0) \to G$ is surjective and $H \subseteq G$, hence $H = G$ and $T' = T$. $\square$

## 4.3 Nori's and Grothendieck's fundamental groups

For the rest of this section suppose that $X$ is connected and reduced, and consider an algebraically closed field $\Omega$ containing $k$. We will regard the rational point $x_0$ as a geometric point $x_0 \in X(\Omega)$.

### 4.3.1 The étale fundamental group

Call $\mathcal{E}(X)$ the category of étale coverings of $X$, and $\omega : \mathcal{E}(X) \to \mathrm{Set}$ the functor sending an étale covering $E \to X$ to the set of $\Omega$-rational points of $E$ over $x_0$, $\omega$ is called the fibre functor.

**Definition 4.14.** The *étale fundamental group* $\pi_1^E(X, x_0)$ is the group of automorphisms of the fibre functor $\omega$.

In order to develop a theory similar to the one of Galois extensions we have studied at the end of Chapter 2 (in fact, we are generalizing it) we would like to define a structure of profinite group-scheme on $\pi_1^E(X, x_0)$: this can be done using Galois coverings. Call $\mathcal{EG}(X)_{x_0}$ the category of pairs $(E, e_0)$ where $E$ is an étale Galois covering and $e_0$ is geometric point in $E_{x_0}$.

**Lemma 4.15.** *Étale Galois coverings are cofinal in $\mathcal{E}(X)$.*

*Proof.* If $E \to X$ is an étale covering and $e_0$ is a geometric point over $x_0$, there exists an étale Galois covering $E' \to X$ with a geometric point $e_0'$ over $x_0$ and a morphism $(E, e_0) \to (E', e_0')$ over $X$ [Mur67, Lemma 4.4.1.8]. □

**Lemma 4.16.** *If $(E, e_0)$, $(E', e_0')$ are pointed étale coverings and $E'$ is connected, there exists at most one morphism $(E', e_0') \to (E, e_0)$ over $X$.*

*Proof.* [Mur67, Lemma 4.4.1.4(**), Lemma 4.4.1.6(i)]. □

**Corollary 4.17.** *The category of pointed étale Galois coverings $\mathcal{EG}(X)_{x_0}$ is cofiltered.*

*Proof.*

- $\mathcal{EG}(X)_{x_0}$ is nonempty, $(X, x_0) \in \mathcal{EG}(X)_{x_0}$.
- If we have two objects $(E_1, e_{1,0})$ and $(E_2, e_{2,0})$, thanks to Lemma 4.15, there exists an étale Galois covering $E$ with a morphism of étale coverings $(E, e_0) \to (E_1 \times_X E_2, e_{1,0} \times e_{2,0})$.
- Thanks to Lemma 4.16, morphisms of pointed étale Galois coverings are unique.

□

If $(E, e_0)$ is an object of $\mathcal{EG}(X)_{x_0}$, the map $g \mapsto ge_0$ gives a bijection $\mathrm{Aut}(E/X) \simeq E_{x_0}$. A morphism $(E', e_0') \to (E, e_0)$ hence induces the composition
$$\mathrm{Aut}(E'/X) \simeq E'_{x_0} \to E_{x_0} \simeq \mathrm{Aut}(E'/X)$$
which is easily checked to be an homomorphism of groups, defining a functor $(E, e_0) \mapsto \mathrm{Aut}(E/X)$. Call $\pi$ the limit of $\mathcal{EG}(X)_{x_0} \to \mathrm{Grp}$, a point of $\pi$ is a family of automorphisms $\lambda_{(E,e_0)} : E \to E$ for every object $(E, e_0)$ in $\mathcal{EG}(X)_{x_0}$ such that for every morphism $(E, e_0) \to (E', e_0')$ the following diagram is commutative
$$\begin{array}{ccc} E & \xrightarrow{\lambda_{(E,e_0)}} & E \\ \downarrow & & \downarrow \\ E' & \xrightarrow{\lambda_{(E',e_0')}} & E' \end{array}$$

This shows that an element of $\pi$ induces an automorphism of the fibre functor

$$\omega' : \mathcal{EG}(X)_{x_0} \to \mathcal{E}(X) \xrightarrow{\omega} \mathrm{Set}$$

by the action of $\mathrm{Aut}(E/X)$ on $\omega(E)$, defining an homomorphism $\pi \to \mathrm{Aut}(\omega')$. Clearly, also an element of $\pi_1^E(X, x_0)$ induces by composition an automorphism of the fibre functor $\omega'$, hence we have another homomorphism $\pi_1^E(X, x_0) = \mathrm{Aut}(\omega) \to \mathrm{Aut}(\omega')$.

**Proposition 4.18.** *The homomorphisms $\pi \to \mathrm{Aut}(\omega')$, $\pi_1^E(X, x_0) \to \mathrm{Aut}(\omega')$ are isomorphisms.*

*Proof.* This is a consequence of [Mur67, Lemma 4.4.1.10]. □

### 4.3.2 Comparison with Nori's fundamental group-scheme

We have found that $\pi_1^E(X, x_0)$ can be seen as a projective limit of finite groups, thus we may regard it a profinite group-scheme thanks to Proposition 2.52. Now we can compare it with $\pi_1^N(X, x_0)$. For the rest of this section, suppose that $k$ is algebraically closed and $\Omega = k$.

Let $\mathcal{ET}(X)_{x_0} \subseteq \mathcal{FT}(X)_{x_0}$ be the category of triples $(T, G, t_0)$ with $T \to X$ finite étale torsor. We have seen in Proposition 3.48 that $(T, G, t_0)$ is an object of $\mathcal{ET}(X)_{x_0}$ if and only if $G$ is finite étale. If $(E, e_0)$ is a pointed étale Galois covering, $(E, \mathrm{Aut}(E/X), e_0)$ where $\mathrm{Aut}(E/X)$ has the structure of discrete group-scheme is an object of $\mathcal{ET}(X)_{x_0}$. We have thus defined an embedding of categories $\mathcal{EG}(X)_{x_0} \subseteq \mathcal{ET}(X)_{x_0}$.

**Lemma 4.19.** *The limits of $\mathcal{EG}(X)_{x_0} \to \mathrm{AffGrp}_k$ and $\mathcal{ET}(X)_{x_0} \to \mathrm{AffGrp}_k$ are canonically isomorphic.*

*Proof.* The forgetful functor $\mathcal{EG}(X)_{x_0} \to \mathrm{AffGrp}_k$ is equal to the composition $\mathcal{EG}(X)_{x_0} \subseteq \mathcal{ET}(X)_{x_0} \to \mathrm{AffGrp}_k$, hence it is enough to show that $\mathcal{EG}(X)_{x_0}$ is cofinal in $\mathcal{ET}(X)_{x_0}$. This is true thanks to Lemma 4.15. □

**Proposition 4.20.** *If $k = \bar{k}$, there is a natural transformation $\pi_1^N \to \pi_1^E$. When $k$ has characteristic 0, this natural transformation is an equivalence of functors.*

*Proof.* Fix a morphism $f : (X, x_0) \to (Y, y_0)$.

We have a commutative diagram of cofiltered categories

$$\begin{tikzcd} \mathcal{ET}(X)_{x_0} \arrow[d] & \mathcal{ET}(Y)_{y_0} \arrow[l, "f^*"'] \arrow[d] \\ \mathcal{FT}(X)_{x_0} & \mathcal{FT}(X)_{y_0} \arrow[l, "f^*"'] \end{tikzcd}$$

Taking projective limits, this gives a commutative diagram

$$\begin{tikzcd} \pi_1^E(X, x_0) \arrow[r, "f_*"] & \pi_1^E(Y, y_0) \\ \pi_1^N(X, x_0) \arrow[u] \arrow[r, "f_*"] & \pi_1^N(Y, y_0) \arrow[u] \end{tikzcd}$$

When $\operatorname{char} k = 0$, $\mathcal{ET}(X)_{x_0} = \mathcal{FT}(X)_{x_0}$ and hence $\pi_1^N = \pi_1^E$. $\square$

# Chapter 5

# Tannakian theory

The main point of the theory is the tannakian interpretation of $\pi_1^N(X, x_0)$, that, under certain hypotheses, will lead us to find the fundamental group-scheme from a particular category of sheaves over $X$. This was done by Nori in [Nor82]; his work has been clarified and extended by Vistoli and Borne in [BV12]. The basic idea is that all the information about an affine group-scheme is contained in the category of its representations. Hence, in this chapter we want to characterize what are the properties of a category of representations, and to find a way to recover the group-scheme from this category. In a more abstract language, we are going to define what a neutral tannakian category is, and to show that the functor sending a group to its category of representations is an equivalence between the category of affine group-schemes and the category of neutral tannakian categories. Most of the content of this chapter comes from [Saa72], [Del82] and [Del90].

## 5.1 Tensor structures

### 5.1.1 Tensor categories

In this first section, we want to introduce a tensor product on an abstract category, having in mind the case of $\mathrm{Vect}_k$.

Let $\mathcal{C}$ be a category and

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \ \ (X, Y) \mapsto X \otimes Y$$

a functor. An *associativity constraint* $\varphi$ is a functorial isomorphism (i.e. an isomorphism of functors)

$$\varphi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

such that, for all $X, Y, Z, T$, the following diagram (the *pentagon axiom*) is commutative.

$$\begin{tikzcd}
& X \otimes (Y \otimes (Z \otimes T)) \arrow[dl, "1 \otimes \varphi"'] \arrow[dr, "\varphi"] & \\
X \otimes ((Y \otimes Z) \otimes T)) \arrow[d, "\varphi"'] & & (X \otimes Y) \otimes (Z \otimes T) \arrow[d, "\varphi"] \\
(X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "\varphi \otimes 1"'] & & ((X \otimes Y) \otimes Z) \otimes T
\end{tikzcd}$$

A *commutativity constraint* is a functorial isomorphism

$$\psi_{X,Y} : X \otimes Y \to Y \otimes X$$

such that $\psi \circ \psi = \mathrm{id}$. An associativity constraint $\varphi$ and a commutativity constraint $\psi$ are compatible if, for all objects $X, Y, Z$, the following diagram (the *hexagon axiom*) is commutative.

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\varphi"] \arrow[d, "1 \otimes \psi"'] & (X \otimes Y) \otimes Z \arrow[d, "\psi"] \\
X \otimes (Z \otimes Y) \arrow[d, "\varphi"'] & Z \otimes (X \otimes Y) \arrow[d, "\varphi"] \\
(X \otimes Z) \otimes Y \arrow[r, "\psi \otimes 1"'] & (Z \otimes X) \otimes Y
\end{tikzcd}$$

A pair $(\mathbb{1}, l)$ comprising an object $\mathbb{1}$ of $\mathcal{C}$ and a functorial isomorphism $l_X : X \to \mathbb{1} \otimes X$ is an *identity object* of $(\mathcal{C}, \otimes, \varphi, \psi)$ if the following diagrams are commutative

$$\begin{tikzcd}
X \otimes Y \arrow[r, "l"] \arrow[d, equal] & \mathbb{1} \otimes (X \otimes Y) \arrow[d, "\varphi"] \\
X \otimes Y \arrow[r, "l \otimes \mathrm{id}"] & (\mathbb{1} \otimes X) \otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \arrow[r, "l \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes l"] & (\mathbb{1} \otimes X) \otimes Y \arrow[d, "\psi \otimes \mathrm{id}"] \\
X \otimes (\mathbb{1} \otimes Y) \arrow[r, "\varphi"] & (X \otimes \mathbb{1}) \otimes Y
\end{tikzcd}$$

**Lemma 5.1.** *If $(\mathbb{1}, l)$ is an identity object, $X \mapsto \mathbb{1} \otimes X$ is an equivalence of categories.*

*Proof.* This a direct consequence of the following categorical lemma. $\square$

**Lemma 5.2.** *Let $\mathcal{A}$ be a category, $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ a functor and $\mathrm{i}_A : A \to \mathcal{F}(A)$ a functorial isomorphism. Then $\mathcal{F}$ is an equivalence of categories.*

*Proof.* We have that $\mathrm{id} : \mathcal{A} \to \mathcal{A}$ is an inverse of $\mathcal{F}$ because $\mathrm{id} \circ \mathcal{F} = \mathcal{F} \circ \mathrm{id} = \mathcal{F}$ is isomorphic to id using i. $\square$

**Definition 5.3.** A system $(\mathcal{C}, \otimes, \varphi, \psi)$ in which $\varphi$ and $\psi$ are compatible associativity and commutativity constraints is a *tensor category* if there exists an identity object.

**Example 5.4.** The category $\mathrm{Mod}_R$ of modules over a commutative ring $R$ becomes a tensor category with the usual tensor product and the obvious constraints. The pair $(R, (a \mapsto 1 \otimes a))$ is obviously an identity object.

**Proposition 5.5.** *Let $(\mathbb{1}, l)$ be an identity object of the tensor category $(\mathcal{C}, \otimes)$.*

- *The functorial isomorphism $r_X = \psi_{\mathbb{1},X} \circ l_X : X \to \mathbb{1} \otimes X \to X \otimes \mathbb{1}$ makes the following diagrams commute:*

$$\begin{tikzcd} X \otimes Y \arrow[r, "r"] \arrow[d, equal] & (X \otimes Y) \otimes \mathbb{1} \arrow[d, "\varphi^{-1}"] \\ X \otimes Y \arrow[r, "\mathrm{id} \otimes r"] & X \otimes (Y \otimes \mathbb{1}) \end{tikzcd} \qquad \begin{tikzcd} X \otimes Y \arrow[r, "r \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes r"] & (X \otimes \mathbb{1}) \otimes Y \arrow[d, "\varphi^{-1}"] \\ X \otimes (Y \otimes \mathbb{1}) \arrow[r, "\mathrm{id} \otimes \psi"] & X \otimes (\mathbb{1} \otimes Y) \end{tikzcd}$$

- *If $(\mathbb{1}', l')$ is another identity object, there exists a unique isomorphism $a : \mathbb{1} \to \mathbb{1}'$ making the diagram*

$$\begin{tikzcd} \mathbb{1} \arrow[r, "l"] \arrow[d, "a"] & \mathbb{1} \otimes \mathbb{1} \arrow[d, "a \otimes a"] \\ \mathbb{1}' \arrow[r, "l'"] & \mathbb{1}' \otimes \mathbb{1}' \end{tikzcd}$$

  *commute.*

*Proof.* • The following diagram is commutative:

$$\begin{tikzcd}
X \otimes Y \arrow[r, "l"] \arrow[d, equal] \arrow[dr, phantom, "1"] & \mathbb{1} \otimes (X \otimes Y) \arrow[r, "\psi"] \arrow[d, "\varphi"] & (X \otimes Y) \otimes \mathbb{1} \arrow[ddd, "\varphi^{-1}"] \\
X \otimes Y \arrow[r, "l \otimes \mathrm{id}"] \arrow[dd, equal] & (\mathbb{1} \otimes X) \otimes Y \arrow[d, "\psi \otimes \mathrm{id}"] \arrow[r, phantom, "3"] & \\
\arrow[r, phantom, "2"] & (X \otimes \mathbb{1}) \otimes Y \arrow[d, "\varphi^{-1}"] & \\
X \otimes Y \arrow[r, "\mathrm{id} \otimes l"] & X \otimes (\mathbb{1} \otimes Y) \arrow[r, "\psi"] & X \otimes (Y \otimes \mathbb{1})
\end{tikzcd}$$

In fact, 1 and 2 are the conditions for $l$ to be an identity object, and 3 is the hexagon axiom. Hence, the first condition on $r$ is respected. For the second condition, the diagram

$$\begin{tikzcd}
X \otimes Y \arrow[r, "l \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes l"] \arrow[dr, phantom, "1"] & (\mathbb{1} \otimes X) \otimes Y \arrow[r, "\psi \otimes \mathrm{id}"] \arrow[d, "\psi \otimes \mathrm{id}"] \arrow[dr, phantom, "3"] & (X \otimes \mathbb{1}) \otimes Y \arrow[dd, "\varphi^{-1}"] \\
X \otimes (\mathbb{1} \otimes Y) \arrow[r, "\varphi"] \arrow[d, "\mathrm{id} \otimes \psi"] \arrow[dr, phantom, "2"] & (X \otimes \mathbb{1}) \otimes Y \arrow[dr, "\varphi^{-1}"] & \\
X \otimes (Y \otimes 1) \arrow[rr, "\mathrm{id} \otimes \psi"] & & X \otimes (\mathbb{1} \otimes Y)
\end{tikzcd}$$

is commutative because 1 is a condition on $l$ to define an identity, and the commutativity of 2 and 3 is trivial.

• Call $a = r^{-1} \circ l' : \mathbb{1} \to \mathbb{1}' \otimes \mathbb{1} \to \mathbb{1}'$. The following diagram is commutative:

$$
\begin{tikzcd}
\mathbb{1} \arrow[r, "l"] \arrow[d, "l'"] \arrow[dr, phantom, "①"] & \mathbb{1}\otimes\mathbb{1} \arrow[r, equal] \arrow[d, "\mathrm{id}\otimes l'"] \arrow[dr, phantom, "②"] & \mathbb{1}\otimes\mathbb{1} \arrow[d, "l'\otimes l'"] \\
\mathbb{1}'\otimes\mathbb{1} \arrow[r, "l"] \arrow[d, equal] \arrow[dr, phantom, "③"] & \mathbb{1}\otimes(\mathbb{1}'\otimes\mathbb{1}) \arrow[r, "l'\otimes\mathrm{id}"] \arrow[d, "a\otimes\mathrm{id}"] \arrow[dr, phantom, "④"] & (\mathbb{1}'\otimes\mathbb{1})\otimes(\mathbb{1}'\otimes\mathbb{1}) \arrow[d, equal] \\
\mathbb{1}'\otimes\mathbb{1} \arrow[r, "l'"] \arrow[d, "r^{-1}"] \arrow[dr, phantom, "⑤"] & \mathbb{1}'\otimes(\mathbb{1}'\otimes\mathbb{1}) \arrow[r, "r\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes r^{-1}"] \arrow[dr, phantom, "⑥"] & (\mathbb{1}'\otimes\mathbb{1})\otimes(\mathbb{1}'\otimes\mathbb{1}) \arrow[d, "r^{-1}\otimes r^{-1}"] \\
\mathbb{1}' \arrow[r, "l'"] & \mathbb{1}'\otimes\mathbb{1}' \arrow[r, equal] & \mathbb{1}'\otimes\mathbb{1}'
\end{tikzcd}
$$

In fact, ① and ⑤ are commutative thanks to the functoriality of $l$ and $l'$, ② and ⑥ are trivial, ④ is the definition of $a$ and ③ is the following, where $X = \mathbb{1}'\otimes\mathbb{1}$:

$$
\begin{tikzcd}
X \arrow[r, "l"] \arrow[d, "l'"] \arrow[dr, phantom, "⑦"] & \mathbb{1}\otimes X \arrow[r, equal] \arrow[d, "l'"] \arrow[dr, phantom, "⑧"] & \mathbb{1}\otimes X \arrow[d, "l'\otimes\mathrm{id}"] \\
\mathbb{1}'\otimes X \arrow[r, "\mathrm{id}\otimes l"] \arrow[d, equal] & \mathbb{1}'\otimes(\mathbb{1}\otimes X) \arrow[r, "\varphi"] \arrow[d, phantom, "⑨"] & (\mathbb{1}'\otimes\mathbb{1})\otimes X \arrow[d, equal] \\
\mathbb{1}'\otimes X \arrow[r, "l\otimes\mathrm{id}"] & (\mathbb{1}\otimes\mathbb{1}')\otimes X \arrow[r, "\psi\otimes\mathrm{id}"] & (\mathbb{1}'\otimes\mathbb{1})\otimes X
\end{tikzcd}
$$

Finally, this is commutative because ⑦ commutes thanks to the functoriality of $l'$, ⑧ thanks to the first condition on $l'$ and ⑨ thanks to the second condition on $l$.

To prove uniqueness of $a$, it is enough to suppose $(\mathbb{1}, l) = (\mathbb{1}', l')$ and prove $a = \mathrm{id}$. We have two commutative diagrams

$$
\begin{tikzcd}
\mathbb{1} \arrow[r, "l"] \arrow[d, "a"] & \mathbb{1}\otimes\mathbb{1} \arrow[d, "a\otimes a"] \\
\mathbb{1} \arrow[r, "l"] & \mathbb{1}\otimes\mathbb{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "l"] \arrow[d, "a^{-1}"] & \mathbb{1}\otimes\mathbb{1} \arrow[d, "\mathrm{id}\otimes a^{-1}"] \\
\mathbb{1} \arrow[r, "l"] & \mathbb{1}\otimes\mathbb{1}
\end{tikzcd}
$$

where the first one is given by hypothesis and the second one comes from functoriality of $l$. The composition of the first one with the second gives

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{l} & \mathbb{1}\otimes\mathbb{1} \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle a\otimes\mathrm{id}} \\ \mathbb{1} & \xrightarrow{l} & \mathbb{1}\otimes\mathbb{1} \end{array}$$

Now, the fact that $l$ is an isomorphism implies $a \otimes \mathrm{id} = \mathrm{id} \otimes \mathrm{id}$, and this in turn implies $a = \mathrm{id}$ because $X \mapsto X \otimes \mathbb{1}$ is an equivalence of categories thanks to the functorial isomorphism $r_X : X \to X \otimes \mathbb{1}$ and Lemma 5.2.

□

### 5.1.2 Abelian tensor categories

**Definition 5.6.** An *abelian tensor category* is an abelian category $\mathcal{C}$ with a structure of tensor category $(\mathcal{C}, \otimes)$ such that $\otimes$ is biadditive.

If $(\mathcal{C}, \otimes)$ is an abelian tensor category, $R = \mathrm{End}(\mathbb{1})$ is a ring with composition. If $(\mathbb{1}', l')$ is a second identity object, the unique isomorphism of Proposition 5.5 defines a unique isomorphism $R \simeq \mathrm{End}(\mathbb{1}')$.

**Lemma 5.7.** *$R$ is commutative.*

*Proof.* Let $a, b \in R$ be endomorphisms of $\mathbb{1}$, we want to show that $a \circ b = b \circ a$. We have that $a \circ b$ is equal to the composition

$$\mathbb{1} \xrightarrow{l} \mathbb{1}\otimes\mathbb{1} \xrightarrow{\mathrm{id}\otimes b} \mathbb{1}\otimes\mathbb{1} \xrightarrow{\mathrm{id}\otimes a} \mathbb{1}\otimes\mathbb{1} \xrightarrow{l^{-1}} \mathbb{1}$$

thanks to functoriality of $l$. Now, since $X \mapsto X \otimes \mathbb{1}$ is an equivalence of categories, there exists an $a' \in \mathrm{End}(\mathbb{1})$ such that $\mathrm{id} \otimes a = a' \otimes \mathrm{id} : \mathbb{1}\otimes\mathbb{1} \to \mathbb{1}\otimes\mathbb{1}$. This implies that $a \circ b$ is the composition

$$1 \xrightarrow{l} \mathbb{1}\otimes\mathbb{1} \xrightarrow{\mathrm{id}\otimes b} \mathbb{1}\otimes\mathbb{1} \xrightarrow{a'\otimes\mathrm{id}} \mathbb{1}\otimes\mathbb{1} \xrightarrow{l^{-1}} \mathbb{1}$$

which is equal to

$$1 \xrightarrow{l} \mathbb{1}\otimes\mathbb{1} \xrightarrow{a'\otimes\mathrm{id}} \mathbb{1}\otimes\mathbb{1} \xrightarrow{\mathrm{id}\otimes b} \mathbb{1}\otimes\mathbb{1} \xrightarrow{l^{-1}} \mathbb{1}$$

and finally

$$1 \xrightarrow{l} \mathbb{1}\otimes\mathbb{1} \xrightarrow{\mathrm{id}\otimes a} \mathbb{1}\otimes\mathbb{1} \xrightarrow{\mathrm{id}\otimes b} \mathbb{1}\otimes\mathbb{1} \xrightarrow{l^{-1}} \mathbb{1},$$

which is $b \circ a$. □

**Lemma 5.8.** *The category $\mathcal{C}$ is $R$-linear and $\otimes$ is $R$-bilinear.*

*Proof.* An element $a \in R$ acts on each object $X$ with the composition

$$X \xrightarrow{l} \mathbb{1} \otimes X \xrightarrow{a \otimes \mathrm{id}} \mathbb{1} \otimes X \xrightarrow{l^{-1}} X.$$

If $X, Y$ are objects, this action on either $X$ or $Y$ induces an action of $R$ on $\mathrm{Hom}(X, Y)$. It is not important if $R$ acts on $X$ or $Y$ because the diagram

$$\begin{tikzcd}
& \mathbb{1} \otimes X \arrow[r, "a\otimes\mathrm{id}"] & \mathbb{1} \otimes X \arrow[r, "\mathrm{id}\otimes f"] & \mathbb{1} \otimes Y \arrow[dr, bend left, "l^{-1}"] & \\
X \arrow[ur, bend left, "l"] \arrow[dr, bend right, "l"'] & & & & Y \\
& \mathbb{1} \otimes X \arrow[r, "\mathrm{id}\otimes f"] & \mathbb{1} \otimes Y \arrow[r, "a\otimes\mathrm{id}"] & \mathbb{1} \otimes Y \arrow[ur, bend right, "l^{-1}"'] &
\end{tikzcd}$$

commutes for every $a \in R$, $f \in \mathrm{Hom}(X, Y)$. This action defines a structure of $R$-module on $\mathrm{Hom}(X, Y)$: if $f, g \in \mathrm{Hom}(X, Y)$ and $a, b \in R$,

$$(a + b) \otimes (f + g) = (a \otimes f) + (a \otimes g) + (b \otimes f) + (b \otimes g)$$

because $\otimes$ is biadditive. Moreover, the composition

$$\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

is $R$-bilinear because $\circ$ is biadditive and we have seen that it is indifferent if $R$ acts on $X$, $Y$ or $Z$. Finally, $\otimes$ is $R$-bilinear because it is biadditive and the action of $R$ on $X \otimes Y$ is the same if we act on $X$, $Y$ or directly on $X \otimes Y$ thanks to the axioms of $l$: the diagram

$$\begin{tikzcd}
X \otimes Y \arrow[r, equal] \arrow[d, "l"] & X \otimes Y \arrow[r, equal] \arrow[d, "l\otimes\mathrm{id}"] & X \otimes Y \arrow[d, "\mathrm{id}\otimes l"] \\
\mathbb{1} \otimes (X \otimes Y) \arrow[r, "\varphi"] & (\mathbb{1} \otimes X) \otimes Y \arrow[r, "\varphi^{-1}\circ(\psi\otimes\mathrm{id})"] & X \otimes (\mathbb{1} \otimes Y)
\end{tikzcd}$$

commutes. □

**Definition 5.9.** An *$R$-linear tensor category* is an abelian tensor category such that $R = \mathrm{End}(\mathbb{1})$.

### 5.1.3 Rigid tensor categories

Let $(\mathcal{C}, \otimes)$ be a tensor category.

Consider $X$ and $Y$ two objects of $\mathcal{C}$, and suppose that there exist morphisms $\delta : \mathbb{1} \to Y \otimes X$ and $\mathrm{ev} : X \otimes Y \to \mathbb{1}$ such that the two compositions

$$X \xrightarrow{\mathrm{id}\otimes\delta} X \otimes Y \otimes X \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} X$$

$$Y \xrightarrow{\delta\otimes\mathrm{id}} Y \otimes X \otimes Y \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} Y$$

are identities. We will call two such morphisms a *duality* between $X$ and $Y$ and say that $Y$ is a *dual* of $X$.

**Definition 5.10.** A tensor category $(\mathcal{C}, \otimes)$ is *rigid* if every object $X$ has a dual.

For every object $X$, call $\mathrm{s}_X$ the functor $T \mapsto T \otimes X$.

**Proposition 5.11.** *Let $Y$ be a dual of $X$. Then $\mathrm{s}_X$ and $\mathrm{s}_Y$ are right adjoints to each other, i.e. there exist bijections $\mathrm{Hom}(S \otimes X, T) \simeq \mathrm{Hom}(S, T \otimes Y)$ and $\mathrm{Hom}(S \otimes Y, T) \simeq \mathrm{Hom}(S, T \otimes X)$ functorial in $S, T$.*

*Proof.* Since everything is symmetrical in $X, Y$, we may restrict ourselves to prove that $\mathrm{s}_Y$ is a right adjoint of $\mathrm{s}_X$.

Hence, take a morphism $f : S \otimes X \to T$ and consider the composition (we omit associativity and commutativity morphisms)

$$S \xrightarrow{\mathrm{id}\otimes\delta} S \otimes X \otimes Y \xrightarrow{f\otimes\mathrm{id}} T \otimes Y$$

which is a morphism in $\mathrm{Hom}(S, T \otimes Y)$. On the other hand, take a morphism $g : S \to T \otimes Y$ and consider the composition

$$S \otimes X \xrightarrow{g\otimes\mathrm{id}} T \otimes Y \otimes X \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} T.$$

The constraints on $\delta$ and ev imply that these two constructions are inverses to each other: if we have a morphism $f : S \otimes X \to T$, then

$$S \otimes X \xrightarrow{\mathrm{id}\otimes\delta\otimes\mathrm{id}} S \otimes X \otimes Y \otimes X \xrightarrow{f\otimes\mathrm{id}\otimes\mathrm{id}} T \otimes Y \otimes X \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} T$$

is equal to

$$S \otimes X \xrightarrow{\mathrm{id}\otimes\delta\otimes\mathrm{id}} S \otimes X \otimes Y \otimes X \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{ev}} S \otimes X \xrightarrow{f} T$$

and

$$S \otimes X \xrightarrow{\mathrm{id} \otimes \delta \otimes \mathrm{id}} S \otimes X \otimes Y \otimes X \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ev}} S \otimes X$$

is the identity of $S \otimes X$. If we start from a morphism $g : S \to T \otimes Y$, the verification is completely analogous. Finally, the functoriality in $S$ and $T$ is obvious from the construction. $\square$

**Corollary 5.12.** *Let $(\mathcal{C}, \otimes)$ be a rigid tensor category. Then $\otimes$ commutes with limits and colimits in each variable; in particular, if $(\mathcal{C}, \otimes)$ is abelian, $\otimes$ is exact.*

*Proof.* For a fixed object $X$, $\mathrm{s}_X$ has a left and right adjoint, $\mathrm{s}_{X^\vee}$, hence we may apply Proposition 1.25. $\square$

**Proposition 5.13.** *Let $(C, \otimes)$ be a rigid tensor category, and $X, Y, Z$ objects of $\mathcal{C}$. Then*

*(i) If $Y$, $Z$ are duals of $X$, there exists a unique morphism $\alpha : Y \to Z$ respecting* ev *and $\delta$, and it is an isomorphism. We will write $X^\vee$ for the dual of $X$.*

*(ii) $X^\vee \otimes Y$ represents the functor $T \mapsto \mathrm{Hom}(T \otimes X, Y)$.*

*(iii) $X \simeq X^{\vee\vee}$.*

*(iv) $X^\vee \otimes Y^\vee \simeq (X \otimes Y)^\vee$.*

*(v) The association $X \mapsto X^\vee$ extends to an equivalence of categories $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$.*

*Proof.* (i) We have that $\mathrm{s}_Y$ and $\mathrm{s}_Z$ are both right adjoints of $\mathrm{s}_X$, hence we obtain a bijection $\mathrm{Hom}(S, \mathrm{s}_Y(T)) \xrightarrow{\sim} \mathrm{Hom}(S, \mathrm{s}_Z(T))$ for every $S, T$ such that the diagram

$$\begin{tikzcd}
\mathrm{Hom}(S, \mathrm{s}_Y(T)) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \mathrm{Hom}(S, \mathrm{s}_Z(T)) \arrow[dl, "\sim"] \\
& \mathrm{Hom}(\mathrm{s}_X(S), T) &
\end{tikzcd}$$

commutes. Since the bijection is functorial in $S$ and $T$, it yields to an isomorphism of functors $\mathrm{s}_Y \xrightarrow{\sim} \mathrm{s}_Z$ thanks to the Yoneda Lemma. Consider now the composition

$$\alpha : Y \xrightarrow{\mathrm{id} \otimes \delta_Z} Y \otimes X \otimes Z \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{id}} Z.$$

It can be seen, following the constructions of Proposition 5.11, that for every object $T$ the isomorphism $\mathrm{s}_Y(T) \to \mathrm{s}_Z(T)$ is exactly $\alpha \otimes \mathrm{id}_T$. In particular, for $T = \mathbb{1}$, we get that $\alpha$ is an isomorphism. Moreover, the commutative diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\delta_Z"] \arrow[d, "\delta_Y"] & Z \otimes X \arrow[r, "\mathrm{id} \otimes \delta_Y \otimes \mathrm{id}"] \arrow[drr, "\mathrm{id}"] & & Z \otimes X \otimes Y \otimes X \arrow[d, "\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ev}_Y"] \\
Y \otimes X \arrow[r, "\mathrm{id} \otimes \delta_Z \otimes \mathrm{id}"'] & Y \otimes X \otimes Z \otimes X \arrow[rr, "\mathrm{ev}_Y \otimes \mathrm{id} \otimes \mathrm{id}"'] & & Z \otimes X
\end{tikzcd}$$

shows that $(\alpha \otimes \mathrm{id}) \circ \delta_Y = \delta_Z$, and the commutative diagram

$$\begin{tikzcd}
Y \otimes X \arrow[r, "\mathrm{id} \otimes \delta_Z \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \delta_Z \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id}"] & Y \otimes X \otimes Z \otimes X \arrow[r, "\mathrm{ev}_Y \otimes \mathrm{id} \otimes \mathrm{id}"] & Z \otimes X \arrow[d, "\mathrm{ev}_Z"] \\
Y \otimes X \otimes Z \otimes X \arrow[r, "\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ev}_Z"'] & Y \otimes X \arrow[r, "\mathrm{ev}_Y"'] & \mathbb{1}
\end{tikzcd}$$

shows that $\mathrm{ev}_Z \circ (\alpha \otimes \mathrm{id}_X) = \mathrm{ev}_Y$.

On the other hand, if $\beta : Y \to Z$ respects $\delta$ and ev, $\mathrm{id} \otimes \beta : T \otimes Y \to T \otimes Z$ clearly gives a morphism of functors $\mathrm{s}_Y \to \mathrm{s}_Z$ such that the diagram

$$\begin{tikzcd}
\mathrm{Hom}(S, \mathrm{s}_Y(T)) \arrow[rr] \arrow[dr, "\sim"] & & \mathrm{Hom}(S, \mathrm{s}_Z(T)) \arrow[dl, "\sim"'] \\
& \mathrm{Hom}(\mathrm{s}_X(S), T) &
\end{tikzcd}$$

commutes. But this implies that $\mathrm{Hom}(S, \mathrm{s}_Y(T)) \to \mathrm{Hom}(S, \mathrm{s}_Z(T))$ is the same map induced by $\alpha$, and hence $\alpha = \beta$ thanks to the Yoneda Lemma.

(ii) The functor $T \mapsto \mathrm{Hom}(T \otimes X, Y)$ is isomorphic to the functor $T \mapsto \mathrm{Hom}(T, X^\vee \otimes Y)$ thanks to Proposition 5.11.

(iii) The definition of the dual is symmetric in $X$ and $X^\vee$, hence $X$ is a dual of $X^\vee$.

(iv) The morphisms

$$\delta_X \otimes \delta_Y : \mathbb{1} \to X \otimes X^\vee \otimes Y \otimes Y^\vee$$

and

$$\mathrm{ev}_X \otimes \mathrm{ev}_Y : X \otimes X^\vee \otimes Y \otimes Y^\vee \to \mathbb{1}$$

define a duality between $X \otimes Y$ and $X^\vee \otimes Y^\vee$.

(v) We have a functorial bijection $\mathrm{Hom}(X, Y) \simeq \mathrm{Hom}(Y^\vee, X^\vee)$, hence $X \to X^\vee$ defines a fully faithful functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$. But $X \simeq X^{\vee\vee}$, hence the functor is essentially surjective, too. If $f : X \to Y$ is a morphism, we will write ${}^t f$ for the corresponding morphism $Y^\vee \to X^\vee$ and call it the *transpose* of $f$.

□

**Example 5.14.** Consider the tensor category $\mathrm{Mod}_R$ of $R$-modules. We claim that a finitely generated module $M$ has a dual if and only if it is projective.

If $M$ has a dual $M^\vee$, the functor $N \mapsto \mathrm{Hom}_R(M, N)$ is isomorphic to $N \mapsto N \otimes M^\vee$ which is right exact, hence $M$ is projective.

We will show the other implication supposing $M$ free, and then we will generalize. If $M$ is free, choose a basis $m_1, \dots, m_n$ and call $m_1^\vee, \dots, m_n^\vee$ the dual basis of $\mathrm{Hom}_R(M, R)$. Define $\delta : R \to M \otimes \mathrm{Hom}_R(M, R)$ sending $1 \mapsto \sum_i m_i \otimes m_i^\vee$, it is easy to check that $\delta$ does not depend on the chosen basis. As ev, take the evaluation $m \otimes f \mapsto f(m)$, a brief calculation shows that $\delta$ and ev define a duality between $M$ and $\mathrm{Hom}_R(M, R)$.

Now, if $M$ is projective, $\widetilde{M}$ is a locally free sheaf on $\operatorname{Spec} R$. The fact that the definition of $\delta$ for free modules does not depend on the basis implies that $\widetilde{\delta} : \mathcal{O}_{\operatorname{Spec} R} \to \widetilde{M} \otimes \underline{\mathrm{Hom}}(\widetilde{M}, \mathcal{O}_{\operatorname{Spec} R})$ is well defined, and $\widetilde{\delta}$ corresponds to an homomorphism $\delta : R \to M \otimes \mathrm{Hom}_R(M, R)$. We can also define ev as above, and the fact that $\delta$ and ev define a duality can be checked at the level of sheaves over $\operatorname{Spec} R$.

### 5.1.4 Tensor functors

Let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be tensor categories, with respective identities $(\mathbb{1}, l)$ and $(\mathbb{1}', l')$.

**Definition 5.15.** A *tensor functor* $(\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ is a pair $(\mathcal{F}, c)$ where $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is a functor and $c_{X,Y} : \mathcal{F}X \otimes' \mathcal{F}Y \xrightarrow{\sim} \mathcal{F}(X \otimes Y)$ is an isomorphism of functors $\otimes' \circ (\mathcal{F} \times \mathcal{F}) \xrightarrow{\sim} \mathcal{F} \circ \otimes$ respecting associativity, commutativity and identity. More precisely, there exists an isomorphism $a : \mathcal{F}\mathbb{1} \xrightarrow{\sim} \mathbb{1}'$, where $(\mathbb{1}', l')$ is an identity of $\mathcal{C}'$, such that the diagrams

$$\begin{tikzcd}
\mathcal{F}\mathbb{1} \arrow[r, "\mathcal{F}l"] \arrow[d, "a"] & \mathcal{F}(\mathbb{1} \otimes \mathbb{1}) \arrow[r, "c^{-1}"] & \mathcal{F}\mathbb{1} \otimes \mathcal{F}\mathbb{1} \arrow[d, "a \otimes a"] \\
\mathbb{1}' \arrow[rr, "l'"] & & \mathbb{1}' \otimes \mathbb{1}'
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}X \otimes' (\mathcal{F}Y \otimes' \mathcal{F}Z) \arrow[r, "\mathrm{id} \otimes c"] \arrow[d, "\varphi'"] & \mathcal{F}X \otimes' \mathcal{F}(Y \otimes Z) \arrow[r, "c"] & \mathcal{F}(X \otimes (Y \otimes Z)) \arrow[d, "\mathcal{F}(\varphi)"] \\
(\mathcal{F}X \otimes' \mathcal{F}Y) \otimes' \mathcal{F}Z \arrow[r, "c \otimes \mathrm{id}"] & \mathcal{F}(X \otimes Y) \otimes' \mathcal{F}Z \arrow[r, "c"] & \mathcal{F}((X \otimes Y) \otimes Z)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}X \otimes' \mathcal{F}Y \arrow[r, "c"] \arrow[d, "\psi'"] & \mathcal{F}(X \otimes Y) \arrow[d, "\mathcal{F}(\psi)"] \\
\mathcal{F}Y \otimes' \mathcal{F}X \arrow[r, "c"] & \mathcal{F}(Y \otimes X)
\end{tikzcd}$$

commute.

*Remark* 5.16. If the isomorphism $a : \mathcal{F}\mathbb{1} \xrightarrow{\sim} \mathbb{1}'$ exists, $\mathcal{F}(\mathbb{1})$ is an identity with an opportune functorial isomorphism $X \to \mathcal{F}\mathbb{1} \otimes X$ and hence $a$ is unique thanks to Proposition 5.5.

**Proposition 5.17.** *Let $(\mathcal{F}, c) : (\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ be a tensor functor between rigid tensor categories. Then, there is a canonical isomorphism $\mathcal{F}(X^\vee) \simeq \mathcal{F}(X)^\vee$.*

*Proof.* If $\delta$ and ev define a duality between $X$ and $X^\vee$, $c^{-1} \circ \mathcal{F}(\delta)$ and $\mathcal{F}(\mathrm{ev}) \circ c$ define a duality between $\mathcal{F}(X)$ and $\mathcal{F}(X^\vee)$. $\square$

### 5.1.5 Morphisms of tensor functors

**Definition 5.18.** Let $(\mathcal{F}, c), (\mathcal{G}, d) : (\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ be tensor functors. A morphism of tensor functors $\lambda : \mathcal{F} \to \mathcal{G}$ is a natural transformation respecting the identity and the tensor product, i.e. for every $X, Y$ in $\mathcal{C}$ the diagrams

$$
\begin{tikzcd}
\mathbb{1}' \arrow[r, "\sim"] \arrow[d, equal] & \mathcal{F}\mathbb{1} \arrow[d, "\lambda"] \\
\mathbb{1}' \arrow[r, "\sim"] & \mathcal{G}\mathbb{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}X \otimes' \mathcal{F}Y \arrow[r, "c"] \arrow[d, "\lambda\otimes\lambda"] & \mathcal{F}(X \otimes Y) \arrow[d, "\lambda"] \\
\mathcal{G}X \otimes' \mathcal{G}Y \arrow[r, "d"] & \mathcal{G}(X \otimes Y)
\end{tikzcd}
$$

commute, where $\mathbb{1}' \xrightarrow{\sim} \mathcal{F}\mathbb{1}$ and $\mathbb{1}' \xrightarrow{\sim} \mathcal{G}\mathbb{1}$ are the unique isomorphisms described in the definition of tensor functors.

**Definition 5.19.** A tensor functor $(\mathcal{F}, c) : (\mathcal{C}, \otimes) \to (\mathcal{C}, \otimes')$ is a *tensor equivalence* if there exists a tensor functor $(\mathcal{G}, d) : (\mathcal{C}', \otimes') \to (\mathcal{C}, \otimes)$ and isomorphisms of tensor functors $\mathcal{F} \circ \mathcal{G} \simeq \mathrm{id}_{\mathcal{C}'}$ and $\mathcal{G} \circ \mathcal{F} \simeq \mathrm{id}_{\mathcal{C}}$.

**Proposition 5.20.** *A tensor functor $(\mathcal{F}, c) : (\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ is a tensor equivalence if and only if $\mathcal{F}$ is an equivalence.*

*Proof.* The "only if" part is obvious. Suppose now that $\mathcal{F}$ is an equivalence of categories. Thanks to Proposition 1.7 and Proposition 1.8, there exists a left adjoint $\mathcal{G} : \mathcal{C}' \to \mathcal{C}$ to $\mathcal{F}$ such that the induced natural transformations $\eta : \mathrm{id}_{\mathcal{C}'} \to \mathcal{F} \circ \mathcal{G}$, $\varepsilon : \mathcal{G} \circ \mathcal{F} \to \mathrm{id}_{\mathcal{C}}$ are isomorphisms of functors and satisfy

$$(\mathcal{F} * \varepsilon) \circ (\eta * \mathcal{F}) = \mathrm{id}_{\mathcal{F}}, \ (\varepsilon * \mathcal{G}) \circ (\mathcal{G} * \eta) = \mathrm{id}_{\mathcal{G}}\,.$$

Now that we have chosen the right quasi-inverse for $\mathcal{F}$, the proof is only a very long verification that everything works. We want to show that there is a way to regard $\mathcal{G}$ as a tensor functor such that $\eta$ and $\varepsilon$ are morphisms of tensor functors.

Given two objects $X'$, $Y'$ in $\mathcal{C}'$, we want to define an isomorphism $d_{X',Y'} : \mathcal{G}X' \otimes \mathcal{G}Y' \xrightarrow{\sim} \mathcal{G}(X' \otimes Y')$. Call $d_{X',Y'}$ the composition

$$\mathcal{G}X' \otimes \mathcal{G}Y' \xrightarrow{\varepsilon^{-1}} \mathcal{G}\mathcal{F}(\mathcal{G}X' \otimes \mathcal{G}Y') \xrightarrow{\mathcal{G}(c^{-1})} \mathcal{G}(\mathcal{F}\mathcal{G}X' \otimes \mathcal{F}\mathcal{G}Y') \xrightarrow{\mathcal{G}(\eta^{-1}\otimes\eta^{-1})} \mathcal{G}(X' \otimes Y')$$

Since $d$ is the composition of functorial isomorphisms, it is a functorial isomorphism, too.

We need to give an isomorphism $a' : \mathcal{G}\mathbb{1}' \to \mathbb{1}$ and check that $(\mathcal{G}, d)$ is a tensor functor. We already have an isomorphism $a : \mathcal{F}\mathbb{1} \to \mathbb{1}'$, define $a'$ as the composition

$$a' : \mathcal{G}\mathbb{1}' \xrightarrow{\mathcal{G}a^{-1}} \mathcal{G}\mathcal{F}\mathbb{1} \xrightarrow{\varepsilon} \mathbb{1}.$$

Firstly, we need to check the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{G}\mathbb{1}' \arrow[r, "\mathcal{G}l'"] \arrow[d, "a'"] & \mathcal{G}(\mathbb{1}' \otimes \mathbb{1}') \arrow[r, "d^{-1}"] & \mathcal{G}\mathbb{1}' \otimes \mathcal{G}\mathbb{1}' \arrow[d, "a' \otimes a'"] \\
\mathbb{1} \arrow[rr, "l"] & & \mathbb{1} \otimes \mathbb{1}
\end{tikzcd}$$

Write it as

$$\begin{tikzcd}
\mathcal{G}\mathbb{1}' \arrow[rr, "\mathcal{G}l'"] \arrow[d, "\mathcal{G}a^{-1}"] \arrow[drr, phantom, "1"] & & \mathcal{G}(\mathbb{1}' \otimes \mathbb{1}') \arrow[r, "d^{-1}"] \arrow[d, "\mathcal{G}(a^{-1} \otimes a^{-1})"'] \arrow[dr, phantom, "2"] & \mathcal{G}\mathbb{1}' \otimes \mathcal{G}\mathbb{1}' \arrow[d, "\mathcal{G}a^{-1} \otimes \mathcal{G}a^{-1}"] \\
\mathcal{G}\mathcal{F}\mathbb{1} \arrow[r, "\mathcal{G}\mathcal{F}l"] \arrow[d, "\varepsilon"] \arrow[drrr, phantom, "3"] & \mathcal{G}\mathcal{F}(\mathbb{1} \otimes \mathbb{1}) \arrow[r, "\mathcal{G}c^{-1}"] & \mathcal{G}(\mathcal{F}\mathbb{1} \otimes \mathcal{F}\mathbb{1}) \arrow[r, "d^{-1}"] & \mathcal{G}\mathcal{F}\mathbb{1} \otimes \mathcal{G}\mathcal{F}\mathbb{1} \arrow[d, "\varepsilon \otimes \varepsilon"] \\
\mathbb{1} \arrow[rrr, "l"] & & & \mathbb{1} \otimes \mathbb{1}
\end{tikzcd}$$

We have that 1 commutes thanks to the condition on $a$ and that 2 commutes thanks to functoriality of $d^{-1}$. For 3, consider the following diagram, where we have expanded the definition of $d^{-1}$:

$$
\begin{tikzcd}
\mathcal{G}\mathcal{F}\mathbb{1} \arrow[r, "\mathcal{G}\mathcal{F}l"] \arrow[d, "\varepsilon"] \arrow[dr, phantom, "\textcircled{4}"] & \mathcal{G}\mathcal{F}(\mathbb{1}\otimes\mathbb{1}) \arrow[r, "\mathcal{G}c^{-1}"] \arrow[d, "\mathcal{G}\mathcal{F}(\varepsilon^{-1}\otimes\varepsilon^{-1})"] \arrow[ddl, "\varepsilon"'] \arrow[dr, phantom, "\textcircled{6}"] & \mathcal{G}(\mathcal{F}\mathbb{1}\otimes\mathcal{F}\mathbb{1}) \arrow[d, "\mathcal{G}(\eta\otimes\eta)"] \\
\mathbb{1} \arrow[d, "l"] & \mathcal{G}\mathcal{F}(\mathcal{G}\mathcal{F}\mathbb{1}\otimes\mathcal{G}\mathcal{F}\mathbb{1}) \arrow[r, "\mathcal{G}c^{-1}"] \arrow[d, "\varepsilon"] \arrow[dl, phantom, "\textcircled{5}"] \arrow[dr, phantom, "\textcircled{7}"] & \mathcal{G}(\mathcal{F}\mathcal{G}\mathcal{F}\mathbb{1}\otimes\mathcal{F}\mathcal{G}\mathcal{F}\mathbb{1}) \arrow[d, "\mathcal{G}c"] \\
\mathbb{1}\otimes\mathbb{1} & \mathcal{G}\mathcal{F}\mathbb{1}\otimes\mathcal{G}\mathcal{F}\mathbb{1} \arrow[l, "\varepsilon\otimes\varepsilon"] & \mathcal{G}\mathcal{F}(\mathcal{G}\mathcal{F}\mathbb{1}\otimes\mathcal{G}\mathcal{F}\mathbb{1}) \arrow[l, "\varepsilon"]
\end{tikzcd}
$$

Commutativity of 4 and 5 descends from functoriality of $\varepsilon$, 7 is trivial and 6 is a consequence of functoriality of $c^{-1}$ and of the fact that

$$\eta\otimes\eta = \mathcal{F}\varepsilon^{-1}\otimes\mathcal{F}\varepsilon^{-1} : \mathcal{F}\mathbb{1}\otimes\mathcal{F}\mathbb{1} \to \mathcal{F}\mathcal{G}\mathcal{F}\mathbb{1}\otimes\mathcal{F}\mathcal{G}\mathcal{F}\mathbb{1}$$

because $(\mathcal{F}*\varepsilon)\circ(\eta*\mathcal{F}) = \mathrm{id}_{\mathcal{F}}$.

We now check the condition on $d$ to respect commutativity of the tensor product. For every $X', Y'$ in $\mathcal{C}'$, we need to check that the diagram

$$
\begin{tikzcd}
\mathcal{G}X'\otimes\mathcal{F}Y' \arrow[r, "d"] \arrow[d, "\psi"] & \mathcal{G}(X'\otimes Y') \arrow[d, "\mathcal{G}(\psi')"] \\
\mathcal{G}Y'\otimes\mathcal{G}X' \arrow[r, "d"] & \mathcal{G}(Y'\otimes X')
\end{tikzcd}
$$

commutes. Write it as

$$
\begin{tikzcd}
\mathcal{G}X'\otimes\mathcal{G}Y' \arrow[r, "\varepsilon"] \arrow[d, "\psi"] \arrow[dr, phantom, "\textcircled{1}"] & \mathcal{G}\mathcal{F}(\mathcal{G}X'\otimes\mathcal{G}Y') \arrow[r, "\mathcal{G}c^{-1}"] \arrow[d, "\mathcal{G}\mathcal{F}\psi"] \arrow[dr, phantom, "\textcircled{2}"] & \mathcal{G}(\mathcal{F}\mathcal{G}X'\otimes\mathcal{F}\mathcal{G}Y') \arrow[r, "\mathcal{G}(\eta^{-1}\otimes\eta^{-1})"] \arrow[d, "\mathcal{G}\psi'"] \arrow[dr, phantom, "\textcircled{3}"] & \mathcal{G}(X'\otimes Y') \arrow[d, "\mathcal{G}\psi'"] \\
\mathcal{G}Y'\otimes\mathcal{G}X' \arrow[r, "\varepsilon"'] & \mathcal{G}\mathcal{F}(\mathcal{G}Y'\otimes\mathcal{G}X') \arrow[r, "\mathcal{G}c^{-1}"'] & \mathcal{G}(\mathcal{F}\mathcal{G}Y'\otimes\mathcal{F}\mathcal{G}X') \arrow[r, "\mathcal{G}(\eta^{-1}\otimes\eta^{-1})"'] & \mathcal{G}(Y'\otimes X')
\end{tikzcd}
$$

Square 1 commutes thanks to functoriality of $\varepsilon$, square 2 thanks to the condition for $c$ to respect commutativity and square 3 thanks to functoriality of $\psi'$. The verification of the condition on associativity is analogous.

We have thus shown that $(\mathcal{G}, d)$ is a tensor functor. We are left with proving that $\eta$ and $\varepsilon$ are morphisms of tensor functors. This is an easy verification, we will do it only for $\eta$: the other case is analogous.

Firstly, we need to check that the diagram

$$\begin{tikzcd} X' \otimes Y' \arrow[r, equal] \arrow[d, "\eta\otimes\eta"] & X' \otimes Y' \arrow[d, "\eta"] \\ \mathcal{F}\mathcal{G}X' \otimes \mathcal{F}\mathcal{G}Y' \arrow[r, "\mathcal{F}d\circ c"] & \mathcal{F}\mathcal{G}(X' \otimes Y') \end{tikzcd}$$

commutes for every $X', Y'$. If we expand the definition of $d$, we obtain:

$$\begin{tikzcd} X' \otimes Y' \arrow[r, "\eta"] \arrow[d, "\eta\otimes\eta"] & \mathcal{F}\mathcal{G}(X' \otimes Y') \\ \mathcal{F}\mathcal{G}X' \otimes \mathcal{F}\mathcal{G}Y' \arrow[d, "c"] & \mathcal{F}\mathcal{G}(\mathcal{F}\mathcal{G}X' \otimes \mathcal{F}\mathcal{G}Y') \arrow[u, "\mathcal{F}\mathcal{G}(\eta\otimes\eta)^{-1}"'] \\ \mathcal{F}(\mathcal{G}X' \otimes \mathcal{G}Y') \arrow[r, "\mathcal{F}\varepsilon^{-1}"] \arrow[uur, "\mathcal{F}d"] & \mathcal{F}\mathcal{G}\mathcal{F}(\mathcal{G}X' \otimes \mathcal{G}Y') \arrow[u, "\mathcal{F}\mathcal{G}c^{-1}"'] \end{tikzcd}$$

This diagram commutes because

$$\mathcal{F}\varepsilon^{-1} = \eta : \mathcal{F}(\mathcal{G}X' \otimes \mathcal{G}Y') \to \mathcal{F}\mathcal{G}\mathcal{F}(\mathcal{G}X' \otimes \mathcal{G}Y')$$

since $(\mathcal{F} * \varepsilon) \circ (\eta * \mathcal{F})$.

Secondly, let $a : \mathcal{F}\mathbb{1} \to \mathbb{1}'$ be the unique isomorphism of identities of $\mathcal{C}'$, we have seen above that

$$a' : \mathcal{G}\mathbb{1}' \xrightarrow{\mathcal{G}a^{-1}} \mathcal{G}\mathcal{F}\mathbb{1} \xrightarrow{\varepsilon} \mathbb{1}$$

is the unique isomorphism of identities $\mathcal{G}\mathbb{1}' \to \mathbb{1}$. Hence, checking that $\eta$ respects identities is equivalent to checking that the composition

$$\mathcal{F}\mathcal{G}\mathbb{1}' \xrightarrow{\mathcal{F}\mathcal{G}a^{-1}} \mathcal{F}\mathcal{G}\mathcal{F}\mathbb{1} \xrightarrow{\mathcal{F}\varepsilon} \mathcal{F}\mathbb{1} \xrightarrow{a} \mathbb{1}'$$

is $\eta_{\mathbb{1}'}^{-1}$. This, in turn, descends from the fact that $(\mathcal{F} * \varepsilon) \circ (\eta * \mathcal{F})$ and hence

$$\mathcal{F}\varepsilon_{\mathbb{1}} = \eta_{\mathcal{F}\mathbb{1}}^{-1} : \mathcal{F}\mathcal{G}\mathcal{F}\mathbb{1} \to \mathcal{F}\mathbb{1}.$$

$\square$

**Proposition 5.21.** *Let $(\mathcal{F}, c), (\mathcal{G}, d) : (\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ be tensor functors. If $\mathcal{C}, \mathcal{C}'$ are rigid, then every morphism of tensor functors $\lambda : \mathcal{F} \to \mathcal{G}$ is an isomorphism.*

*Proof.* The morphism of functors $\mu : \mathcal{G} \to \mathcal{F}$ making the diagrams

$$\begin{tikzcd} \mathcal{F}(X^\vee) \arrow[r, "\lambda_{X^\vee}"] \arrow[d, "\wr"] & \mathcal{G}(X^\vee) \arrow[d, "\wr"] \\ \mathcal{F}(X)^\vee \arrow[r, "{}^t\mu_X"] & \mathcal{G}(X)^\vee \end{tikzcd}$$

commutative for all $X$ in $\mathcal{C}$ is an inverse for $\lambda$. We want to show that the composition

$$\mathcal{F}(X) \xrightarrow{\lambda} \mathcal{G}(X) \xrightarrow{\mu} \mathcal{F}(X)$$

is the identity. We transpose everything and get

$$\mathcal{F}(X)^\vee \xrightarrow{{}^t\mu} \mathcal{G}(X)^\vee \xrightarrow{{}^t\lambda} \mathcal{F}(X)^\vee.$$

To check that this is the identity, consider the diagram

$$\begin{tikzcd} \mathcal{F}(X^\vee) \arrow[r, "\lambda"] \arrow[d, "\wr"] & \mathcal{G}(X^\vee) \arrow[d, "\wr"] & \\ \mathcal{F}(X)^\vee \arrow[r, "{}^t\mu_X"] & \mathcal{G}(X)^\vee \arrow[r, "{}^t\lambda_X"] & \mathcal{F}(X)^\vee \end{tikzcd}$$

Thanks to Proposition 5.13.i, it is enough to show that

$$\mathcal{F}(X^\vee) \xrightarrow{\lambda_{X^\vee}} \mathcal{G}(X^\vee) \xrightarrow{\sim} \mathcal{G}(X)^\vee \xrightarrow{{}^t\lambda} \mathcal{F}(X)^\vee$$

is the only morphism respecting ev and $\delta$, and this is an easy consequence of the fact that $\lambda$ is a tensor functor. The verification of $\lambda \circ \mu = \mathrm{id}_{\mathcal{F}}$ is analogous. □

## 5.2 Neutral Tannakian categories

Fix a field $k$.

**Definition 5.22.** A triple $(\mathcal{C}, \otimes, \omega)$ is a *neutral Tannakian category* over $k$ if $(\mathcal{C}, \otimes)$ is a rigid, $k$-linear tensor category and $\omega : \mathcal{C} \to \mathrm{Vect}_k$ is an exact, faithful and $k$-linear functor. Any such functor is said to be a *fibre functor* for $(\mathcal{C}, \otimes)$. If $(\mathcal{C}', \otimes', \omega')$ is another neutral Tannakian category, a functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is a morphism of neutral Tannakian categories if it is an additive tensor functor such that $\omega' \circ \mathcal{F} = \omega$.

### 5.2.1 Recovering a group from its representations

If $G$ is an affine group-scheme over $k$, the category $\mathrm{Rep}_k G$ of finite dimensional representations of $G$ over $k$ is a rigid abelian tensor category with the usual tensor product, and it becomes a neutral Tannakian category with the forgetful functor $\omega : \mathrm{Rep}_k G \to \mathrm{Vect}_k$.

Let $G$ be an affine group-scheme over $k$, and let $\omega$ be the forgetful functor $\mathrm{Rep}_k G \to \mathrm{Vect}_k$. For a scheme $X$ over $k$, $\underline{\mathrm{Aut}}^{\otimes}(\omega)(X)$ consists of the families $(\lambda_V)$, $V \in \mathrm{Obj}(\mathrm{Rep}_k G)$, where $\lambda_V$ is an $\mathrm{H}^0(X)$-linear automorphism of $V \otimes \mathrm{H}^0(X)$ such that $\lambda_{V_1 \otimes V_2} = \lambda_{V_1} \otimes \lambda_{V_2}$, $\lambda_{\mathbb{1}}$ is the identity and

$$\begin{tikzcd} V \otimes \mathrm{H}^0(X) \arrow[r, "\lambda_V"] \arrow[d, "\alpha\otimes\mathrm{id}"] & V \otimes \mathrm{H}^0(X) \arrow[d, "\alpha\otimes\mathrm{id}"] \\ W \otimes \mathrm{H}^0(X) \arrow[r, "\lambda_W"] & W \otimes \mathrm{H}^0(X) \end{tikzcd}$$

is commutative for every $G$-equivariant map $\alpha : V \to W$. Clearly, $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is a contravariant functor from $\mathrm{Sch}/k$ to $\mathrm{Grp}$. Every $g \in G(X)$ defines a $\mathrm{H}^0(X)$-linear automorphism $g_V$ of $V \otimes \mathrm{H}^0(X)$ for every representation $V$ of $G$, and the conditions for $(g_V)$ to define an element of $\underline{\mathrm{Aut}}^{\otimes}(\omega)(X)$ are trivially satisfied. This defines an homomorphism $G \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$ of functors $\mathrm{Sch}/k^{\mathrm{op}} \to \mathrm{Grp}$.

**Proposition 5.23.** *The natural map $G \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is an isomorphism of functors.*

*Proof.* Let $V \in \mathrm{Rep}_k G$, and call $G_V \subseteq \mathrm{GL}_V$ the image of $G$ in $\mathrm{GL}_V$. Thanks to Lemma 2.46, $\mathrm{Rep}_k G_V \subseteq \mathrm{Rep}_k G$ is the strictly full subcategory of $\mathrm{Rep}_k G$ of objects isomorphic to a subquotient of $p(V, V^{\vee})$, where $p \in \mathbb{N}[t, s]$ is calculated on $V, V^{\vee}$ interpreting sums and multiplications as direct sums and tensor products.

Let $\mathrm{Rep}_k' G \subseteq \mathrm{Rep}_k G$ the wide subcategory with only injective maps. We claim that the map $\lambda \mapsto \lambda_V$ identifies $\underline{\mathrm{Aut}}^{\otimes}(\omega|\,\mathrm{Rep}_k G_V)(X)$ with

$G_V(X) \subseteq \mathrm{GL}(V \otimes \mathrm{H}^0(X))$, and then, passing to the limits along $\mathrm{Rep}'_k G$, $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ thanks to Corollary 2.44.

Since we are going to take limits along $\mathrm{Rep}'_k G$, up to replacing $V$ with $V \oplus V^\vee$ we may suppose that all the elements of $\mathrm{Rep}_k G_V$ are subquotients of $p(V)$ for some $p \in \mathbb{N}[t]$. Hence,

$$\lambda \mapsto \lambda_V : \underline{\mathrm{Aut}}^{\otimes}(\omega|\,\mathrm{Rep}_k G_V)(X) \to \mathrm{GL}(V \otimes \mathrm{H}^0(X))$$

is injective because $\lambda_{p(V)} = p(\lambda_V)$. A posteriori, since we will know that $G_V(X) = \underline{\mathrm{Aut}}^{\otimes}(\omega|\,\mathrm{Rep}_k G_V)(X)$, this remark will be useless: $\lambda$ will be the image of some $g \in G(X)$ and hence $\lambda_{V^\vee} = \lambda_V^\vee$. Anyway, now we need it to ensure injectivity of $\lambda \mapsto \lambda_V$.

Clearly,

$$G_V(X) \subseteq \underline{\mathrm{Aut}}^{\otimes}(\omega|\,\mathrm{Rep}_k G_V)(X) \subseteq \mathrm{GL}(V \otimes \mathrm{H}^0(X)),$$

we want now to prove $\underline{\mathrm{Aut}}^{\otimes}(\omega|\,\mathrm{Rep}_k G_V)(X) \subseteq G_V(X)$.

If $W \in \mathrm{Obj}(\mathrm{Rep}_k G_V)$ and $t \in W \otimes \mathrm{H}^0(X)$ is fixed by $G$, then

$$\alpha : \mathbb{1} \otimes \mathrm{H}^0(X) \to W \otimes \mathrm{H}^0(X)$$

$$1 \otimes a \mapsto at$$

is $G(X)$-equivariant, and so

$$\lambda_W(t) = \lambda_W \alpha(1 \otimes 1) = \alpha \lambda_{\mathbb{1}}(1 \otimes 1) = \alpha(1 \otimes 1) = t.$$

Then, $\underline{\mathrm{Aut}}^{\otimes}(\omega|\,\mathrm{Rep}_k G_V)(X)$ fixes all tensors in representations of $G_V(X)$ fixed by $G_V(X)$, which implies $\underline{\mathrm{Aut}}^{\otimes}(\omega|\,\mathrm{Rep}_k G_V)(X) \subseteq G_V(X)$ thanks to Lemma 2.47. This was the crucial point: in fact, all the rest of the proof is just a limit process to pass from the algebraic case to the general one.

If we have a $G$-equivariant injective map $V \to W$ for some representation $W$ of $G$, we have an induced map $G_W \to G_V$ of restriction, and $\mathrm{Rep}_k G_V \subseteq \mathrm{Rep}_k G_W$ thanks to Lemma 2.46. Moreover, the diagram

$$\begin{tikzcd}
G_W \arrow[r, "\simeq"] \arrow[d] & \underline{\mathrm{Aut}}^{\otimes}(\omega|\,\mathrm{Rep}_k G_W) \arrow[d] \\
G_V \arrow[r, "\simeq"] & \underline{\mathrm{Aut}}^{\otimes}(\omega|\,\mathrm{Rep}_k G_V)
\end{tikzcd}$$

commutes.

Thanks to Corollary 2.44, we already know that

$$G = \varprojlim_{V \in \mathrm{Rep}'_k G} G_V$$

as functors $\mathrm{Sch}\,/k^{\mathrm{op}} \to \mathrm{Grp}$ and so, if we show that

$$\underline{\mathrm{Aut}}^{\otimes}(\omega) = \varprojlim_{V \in \mathrm{Rep}'_k G} \underline{\mathrm{Aut}}^{\otimes}(\omega | \mathrm{Rep}_k G_V),$$

we have finished. But this is obvious since $\mathrm{Rep}_k G = \bigcup_V \mathrm{Rep}_k G_V$. □

A homomorphism $f : G \to G'$ defines a tensor functor $f^* : \mathrm{Rep}_k G' \to \mathrm{Rep}_k G$ such that $\omega^G \circ f^* = \omega^{G'}$. This makes the association $G \mapsto \mathrm{Rep}_k G$ into a functor $\mathrm{Rep}_k$ from $\mathrm{AffGrp}_k^{\mathrm{op}}$ to the category of neutral Tannakian categories $\mathrm{Tan}_k$. Our next result says that this functor is fully faithful.

**Corollary 5.24.** *Let $G$, $H$ be affine group-schemes over $k$, and let $\mathcal{F} : \mathrm{Rep}_k H \to \mathrm{Rep}_k G$ be a morphism of neutral Tannakian categories. Then there exists a unique homomorphism $f : G \to H$ such that $\mathcal{F} \simeq f^*$.*

*Proof.* Such an $\mathcal{F}$ defines an homomorphism of group functors $\mathcal{F}^* : \underline{\mathrm{Aut}}^{\otimes}(\omega^G) \to \underline{\mathrm{Aut}}^{\otimes}(\omega^H)$, hence this defines a unique homomorphism $G \to H$ thanks to the Yoneda Lemma. Obviously $\mathcal{F} \mapsto \mathcal{F}^*$ and $f \mapsto f^*$ are inverse constructions, up to a functorial isomorphism. □

### 5.2.2 The group of a neutral Tannakian category

In Proposition 5.23, we have seen that an affine group-scheme $G$ represents the the functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ of linear automorphisms of the fibre functor $\omega$ on $\mathrm{Rep}_k G$. Now take a generic neutral tannakian category $(\mathcal{C}, \otimes, \omega)$ over $k$ and consider, as before, the functor of linear automorphisms $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ on $\mathcal{C}$.

**Theorem 5.25.** *The functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is represented by an affine group-scheme $G$, and $\omega$ defines a functor $\mathcal{C} \to \mathrm{Rep}_k G$ which is an equivalence of neutral Tannakian categories. As a corollary, the functor $\mathrm{Rep}_k : \mathrm{AffGrp}_k \to \mathrm{Tan}_k$ is an equivalence of categories.*

*Proof.* [Del82, Theorem 2.11]. □

# Chapter 6

# Tannakian interpretation

## 6.1 Representations and vector bundles

Now we turn back to the fundamental group-scheme. Our main concern is to show that the Tannakian category $\mathrm{Rep}_k\, \pi_1^N(X, x_0)$, under certain hypothesis on $X$, is isomorphic to a particular category of vector bundles over $X$, as will be stated in Theorem 6.12.

Let $\varphi : \pi_1^N(X, x_0) \to \mathrm{GL}_V$ a representation; since $\mathrm{GL}_V$ is algebraic, thanks to Proposition 2.60, there is a triple $(T, G, t_0)$ in $\mathcal{T}_{X,x_0}$ such that $\varphi$ splits as

$$\varphi' \circ \mathrm{p}_G : \pi_1^N(X, x_0) \to G \to \mathrm{GL}_V,$$

with $(\mathrm{p}_T, \mathrm{p}_G) : (\widetilde{T}, \pi_1^N(X, x_0), \widetilde{t}_0) \to (T, G, t_0)$ the unique morphism in $\mathcal{T}_{(X,x_0)}$.

Now, as we have seen in Example 3.27, this defines a $G$-equivariant sheaf $\mathcal{O}_T \otimes V$ on $T$. By Theorem 3.29, $\mathcal{O}_T \otimes V$ induces a locally free sheaf $\lambda_V$ on $X$ (the fact that $\mathrm{p}_T : T \to X$ is flat ensures that $\rho$ on $X$ is locally free if and only if $\mathrm{p}_T^* \rho$ is locally free). From now on, by "vector bundle" we will mean "quasi-coherent locally free sheaf of finite rank".

This construction doesn't depend on $(T, G, t_0)$. In fact, take another triple $(T', G', t_0')$ such that $\varphi$ splits as

$$\varphi : \pi_1^N(X, x_0) \to G' \to \mathrm{GL}_V .$$

Without loss of generality, we may suppose there is a morphism $(T', G', t_0') \to (T, G, t_0)$. Since $\pi_1^N(X, x_0)$ is an initial object in $\mathcal{T}_{(X,x_0)}$ we have the splitting $\varphi : \pi_1^N(X, x_0) \to G' \to G \to \mathrm{GL}_V$. Moreover, we have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}_{T'} \otimes V & \longrightarrow & \mathcal{O}_T \otimes V & \longrightarrow & \lambda_V \\
\downarrow & & \downarrow & & \downarrow \\
T' & \xrightarrow{\mathrm{p}_{T,T'}} & T & \xrightarrow{\mathrm{p}_T} & X
\end{array}$$

and

$$\mathrm{p}_{T'}^* \lambda_V \simeq \mathrm{p}_{T,T'}^* \mathrm{p}_T^* \lambda_V \simeq \mathrm{p}_{T,T'}^*(\mathcal{O}_T \otimes V) \simeq \mathcal{O}_{T'} \otimes V.$$

The fact that $\mathrm{p}_{T,T'}$ intertwines the actions of $G$ and $G'$ ensures that the induced action of $G'$ on $\mathcal{O}_{T'} \otimes V \simeq \mathrm{p}_{T'}^* \lambda_V$ is the one we want.

To sum up, we've taken a representation $V$ of $\pi_1^N(X, x_0)$ and we've seen that there exists a torsor $T$ with structure group a quotient $G$ of $\pi_1^N(X, x_0)$ such that $V$ can be regarded as a representation of $G$. The group $G$ acts naturally on the trivial bundle $\mathcal{O}_T \otimes V$, and this induces a vector bundle on $X$ thanks to descent theory.

## 6.2 Full faithfulness of $\mathrm{Rep}_k\, \pi_1 \to \mathrm{Vect}(X)$

We have thus constructed a functor $\Phi$ from the category of representation of $\pi_1^N(X, x_0)$ to that of vector bundles on $X$: we will see that $\Phi$ is fully faithful and then we will describe its essential image. In order to do this, Nori asked $X$ to be proper. We will relax this condition asking $X$ only to be *pseudo-proper*, i.e. $X$ is quasi-compact and, for every vector bundle $E$ on $X$, we ask $\mathrm{H}^0(X, E) < \infty$. We also ask $X$ to be geometrically connected and geometrically reduced. With these hypotheses, the global sections of $\mathcal{O}_X$ are trivial.

**Lemma 6.1.** *$\mathrm{H}^0(X, \mathcal{O}) = k$.*

*Proof.* $\mathrm{H}^0(X, \mathcal{O})$ is a finite ring over $k$, hence it is a product of artinian local rings. But $X$ is connected, hence there is only one factor and $\mathrm{H}^0(X, \mathcal{O})$ is local artinian. Moreover $X$ is reduced, and so $\mathrm{H}^0(X, \mathcal{O})$ is a finite extension of $k$. Since $\mathrm{H}^0(X, \mathcal{O})$ is a finite field over $k$, it is exactly $k$ if and only if $\mathrm{H}^0(X, \mathcal{O}) \otimes \bar{k}$ is reduced and has no nontrivial idempotents.

Now, $\mathrm{H}^0(X_{\bar{k}}, \mathcal{O})$ is reduced and has no nontrivial idempotents because $X$ is geometrically reduced and geometrically connected, hence it is enough to show that the canonical map

$$\mathrm{H}^0(X, \mathcal{O}) \otimes_k \bar{k} \to \mathrm{H}^0(X_{\bar{k}}, \mathcal{O})$$

is injective.

In order to do this, take a finite, affine covering $\{X_i\}$ of $X$ and consider this commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^0(X,\mathcal{O}) \otimes \bar{k} \arrow[r] \arrow[d] & \left(\prod_i \mathrm{H}^0(X_i,\mathcal{O})\right) \otimes \bar{k} \arrow[d] \\
\mathrm{H}^0(X_{\bar{k}},\mathcal{O}) \arrow[r] & \prod_i \left(\mathrm{H}^0(X_{i,\bar{k}},\mathcal{O})\right)
\end{tikzcd}$$

The first row is injective because $\bar{k}$ is flat over $k$, and the last vertical arrow is an isomorphism because the covering is finite and affine. Hence the first vertical arrow is injective, as desired. □

**Corollary 6.2.** *Every morphism from $X$ to an affine $k$-scheme* $\operatorname{Spec} A$ *factors through* $\operatorname{Spec} k$*.*

*Proof.* In general, morphisms $X \to \operatorname{Spec} A$ split as morphisms $X \to \operatorname{Spec} \mathrm{H}^0(X,\mathcal{O}_X) \to \operatorname{Spec} A$. □

**Lemma 6.3.** *Let $\pi_1^N(X,x_0) \to G$ be a finite quotient, and $T \to X$ the associated Nori-reduced $G$-torsor. Then,* $\mathrm{H}^0(T,\mathcal{O}) = k$*.*

*Proof.* Let us call $G = \operatorname{Spec} A$, $C = \mathrm{H}^0(T,\mathcal{O})$ and $U = \operatorname{Spec} C$. We want to prove $U = \operatorname{Spec} k$.

We have a rational point $t_0 \in T(k)$ with image $u_0 \in U(k)$. The action of $G$ on $T$ induces an action on $U$, call $H \subseteq G$ the stabilizer of $u_0$: this defines a morphism $\varphi : G/H = \operatorname{Spec} B \to U$, with $B \subseteq A$.

*Step 1*: $\varphi$ is a closed embedding.

We want to show that $\varphi^\# : C \to B$ is surjective: this is true if and only if $\varphi^\#_{\bar{k}} : C_{\bar{k}} \to B_{\bar{k}}$ is surjective, because $k \to \bar{k}$ is faithfully flat. Hence, we may suppose $k = \bar{k}$.

Consider the following diagram:

$$\begin{tikzcd}
 & G \arrow[dl, "\pi"'] \arrow[dr] & \\
G/H \arrow[rr, "\varphi"] & & U
\end{tikzcd}$$

We have that

$$\pi^{-1}\varphi^{-1}(u_0) = G_{a_0} = H = \pi^{-1}([H]).$$

This means that the natural map $[H] = \operatorname{Spec} k \to \varphi^{-1}(u_0)$ becomes an isomorphism after the pullback along $\pi$. Since $\pi$ is faithfully flat, we conclude that $\varphi^{-1}(u_0) = [H]$.

At the level of global sections, $\varphi^{-1}(u_0) = [H]$ means $B \otimes_C k = k$, where $C \to k$ is the homomorphism defining $u_0$, and $C \to B$ is $\varphi^{\#}$. Let $\mathfrak{p} \subseteq C$ the ideal of $u_0$: we have

$$B_{\mathfrak{p}} \otimes_{C_{\mathfrak{p}}} k = B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = k,$$

where $B_{\mathfrak{p}} = B \otimes_C C_{\mathfrak{p}}$. Now, let $N \subseteq B_{\mathfrak{p}}$ be the image of $C_{\mathfrak{p}}$: since $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = k$, we have $\mathfrak{p}B_{\mathfrak{p}} + N = B_{\mathfrak{p}}$ and hence $N = B_{\mathfrak{p}}$ thanks to Nakayama's lemma.

We have shown that $\varphi^{\#}$ is surjective at $[H]$, now we want to extend this fact to every point of $B$ using the fact that $C \to B$ is $G$-equivariant. Let $B = B_0 \times \cdots \times B_n$, with $B_i$ local and $\mathfrak{m}_i \subseteq B_i$ maximal. We may suppose that $B_0$ is the factor corresponding to the point $[H] \in G/H(k)$, and hence $B_0 \times 0 \cdots \times 0$ is contained in the image of $C \to B$. Now, since $B_i/\mathfrak{m}_i = k$ because $k = \bar{k}$, for every $i$ there exists $g \in G(k)$ defining an automorphism of $B$ such that $g(B_0) = B_i$. We also have that $g$ defines an automorphism of $C$, and the composition

$$C \xrightarrow{g^{-1}} C \xrightarrow{\varphi^{\#}} B \xrightarrow{g} B$$

is $\varphi^{\#}$ because $\varphi^{\#}$ is $G$-equivariant. But this shows that $0 \times \cdots \times B_i \times \cdots \times 0$ is contained in the image of $\varphi^{\#}$, and hence $\varphi^{\#}$ is surjective. Call $V \subseteq U$ the image of $G/H \to U$.

*Step 2*: $V_{\mathrm{red}} = U_{\mathrm{red}}$.

Consider the morphism $T \to X$ as a fpqc covering. Descent theory (Theorem 3.9) tells us that $\mathrm{H}^0(X, \mathcal{O}) \subseteq \mathrm{H}^0(T, \mathcal{O})$ is the subset of sections $s\,\mathrm{H}^0(T, \mathcal{O})$ such that the two restrictions of $s$ to $T \times_X T$ coincide. If we pull back this condition along the isomorphism $G \times T \to T \times_X T$, we see that we are asking $\mathrm{pr}_X^{\#}(s) = \alpha^{\#}(s) \in \mathrm{H}^0(G \times T, \mathcal{O})$, i.e. $k = \mathrm{H}^0(X, \mathcal{O}) = \mathrm{H}^0(T, \mathcal{O})^G = C^G$. Thanks to Theorem 3.45, this means that $(U/G)_{\mathrm{rs}} = \operatorname{Spec} k$ has only one point, and hence $V = U$ set-theoretically.

*Step 3*: $U = V = \operatorname{Spec} k$.

Let $f$ be the natural morphism $T \to U$, $f^{-1}(V) \subseteq T$ is a $G$-invariant closed subscheme such that $f^{-1}(V)_{\mathrm{red}} = T_{\mathrm{red}}$. Let $I \subseteq \mathcal{O}_T$ be the sheaf of ideals defining $f^{-1}(V)$, the fact that $f^{-1}(V)$ is $G$-invariant implies that $I$ inherits from $\mathcal{O}_T$ a structure of $G$-equivariant sheaf. Then, $I^G \subseteq \mathcal{O}_X$ defines a closed subscheme $Y \subseteq X$ such that $Y_{\mathrm{red}} = X_{\mathrm{red}}$: but $X$ is reduced, hence $Y = X$ and so $f^{-1}(V) = T$. Hence, $T \to U$ splits as $T \to V \to U$, but $U = \operatorname{Spec} \mathrm{H}^0(T, \mathcal{O})$ and $V$ is affine, and so $V = U$. Moreover, we get a map $T \to V = G/H$: but $T$ is Nori-reduced, and hence $G = H$ thanks to Proposition 4.11. In particular, $U = V = G/H = \operatorname{Spec} k$. $\square$

**Proposition 6.4.** $\Phi$ *is fully faithful.*

*Proof.* Take two finite representation $V, W$ of $\pi_1^N(X, x_0)$, choose a finite quotient $\pi_1^N(X, x_0) \to G$ such that the actions on both $V$ and $W$ factor through $G$; call $T \to X$ the associated Nori-reduced torsor. Thanks to Corollary 3.30, we have an equivalence $\mathrm{Vect}^G(T) \to \mathrm{Vect}(X)$, hence it is enough to show that $\Psi : \mathrm{Rep}_k G \to \mathrm{Vect}^G(T)$ is fully faithful, i.e.

$$\Psi_{V,W} : \mathrm{Hom}_{\mathrm{Rep}_k G}(V, W) \to \mathrm{Hom}_G(\mathcal{O}_T \otimes V, \mathcal{O}_T \otimes W)$$

is a bijection.

Take a $G$-equivariant morphism $\varphi : V \to W$ and a rational point $t \in T$. We have that $\Psi_{V,W}(\varphi) : \mathcal{O}_T \otimes V \to \mathcal{O}_T \otimes W$, when restricted to the fibers over $t$, is exactly $\varphi$, hence $\Psi_{V,W}$ is injective.

On the other hand, take a $G$-equivariant morphism

$$f : \mathcal{O}_T \otimes V \to \mathcal{O}_T \otimes W$$

of vector bundles, it can be thought as a $G$-equivariant global section $s$ of the trivial vector bundle $\mathcal{O}_T \otimes (V^\vee \otimes W)$. Since $V^\vee \otimes W$ is free and $T$ is Nori-reduced and quasi-compact (because $X$ is quasi-compact and $T \to X$ is affine),

$$\mathrm{H}^0(T, \mathcal{O}_T \otimes (V^\vee \otimes W)) = \mathrm{H}^0(T, \mathcal{O}_T) \otimes (V^\vee \otimes W) = V^\vee \otimes W$$

and so $s$ is a constant, $G$-equivariant global section. This means that $s$ is of the form $1 \otimes v$, with $v \in V^\vee \otimes W$, and $v$ is fixed by $G$. In fact, if $S$ is a scheme, take a point $p \in T(S)$ (for example, the composition of a rational point of $T$ with the structure morphism $S \to \mathrm{Spec}\, k$) and consider the constant section $(p, v = s(p) \in \mathrm{H}^0(S) \otimes V^\vee \otimes W)$:

$$\begin{tikzcd}
\mathcal{O}_S \arrow[r] \arrow[d, mapsto] & \mathcal{O}_T \arrow[r, "s"] \arrow[d, mapsto] & \mathcal{O}_T \otimes V^\vee \otimes W \arrow[d, mapsto] \\
S \arrow[r, "p"] & T \arrow[r] & T
\end{tikzcd}$$

If $g \in G(S)$, $g$ acts on $(p, v)$ as $g \cdot (p, v) = (gp, gv)$. But $s$ is $G$-equivariant, hence

$$(gp, gv) = (gp, gs(p)) = (gp, s(gp)) = (gp, v)$$

because $s$ is constant, and so $gv = v$.

The $G$-invariant vector $v \in V^\vee \otimes W$ defines a $G$-equivariant linear map $f' : V \to W$ such that $\Psi_{V,W}(f') = f$, as desired. $\square$

## 6.3 Essentially finite vector bundles

We have seen that $\Phi$ is fully faithful, now we want to characterize its essential image. As we will show, it consists of vector bundles with a particular finiteness condition, the essentially finite vector bundles.

Let $p \in \mathbb{N}[t]$ be a polynomial and $E$ a vector bundle on $X$, we may define $p(E)$ interpreting sums as direct sums and products as tensor products.

**Definition 6.5.** A vector bundle $E$ is finite if there exist $f$ and $g$ in $\mathbb{N}[t]$ with $f \neq g$ and such that $f(E) \simeq g(E)$.

Now, let $K(X)$ the Grothendieck group associated to the additive monoid $\operatorname{Vect} X$. It is the group of pairs of vector bundles $[V, W]$ over $X$ with the equivalence relation $[V, W] \sim [V', W']$ if $V \oplus W' \simeq V' \oplus W$. The idea is "$[V, W] = V - W$". It has a natural structure of a commutative ring with identity:

$$[V, W] + [V', W'] = [V \oplus V', W \oplus W']$$

$$[V, W] \cdot [V', W'] = [(V \otimes V') \oplus (W \otimes W'), (V \otimes W') \oplus (W \otimes V')].$$

The fact that $X$ is pseudo-proper, as shown in [Ati56], ensures that the Krull-Schmidt-Remak theorem holds, hence $K(X)$ is a free abelian group with basis the set of indecomposable vector bundles over $X$ up to isomorphism.

**Definition 6.6.** For a vector bundle $V$, call $\mathrm{S}(V)$ the set of all indecomposable components of $V^{\otimes n}$ for all non negative integers $n$.

**Lemma 6.7.** *Let $V$ be a vector bundle over $X$. The following are equivalent:*

*(i)* $\mathrm{S}(V)$ *is finite.*

*(ii)* $[V] \in K(X)$ *is integral over* $\mathbb{Z}$.

*(iii)* $V$ *is finite.*

*(iv)* $[V] \otimes 1$ *is algebraic over* $\mathbb{Q}$ *in* $K(X) \otimes \mathbb{Q}$.

*Proof.* Consider the extensions of rings $\mathbb{Z}[V] \subseteq \mathbb{Z}[\mathrm{S}(V)] \subseteq K(X)$, where $\mathbb{Z}[\mathrm{S}(V)] \subseteq K(X)$ is subring generated by $\mathrm{S}(V)$.

$(i) \implies (ii)$ because $[V]$ is in $\mathbb{Z}[\mathrm{S}(V)]$, which is finite over $\mathbb{Z}$.

$(ii) \implies (iii)$ and $(iii) \implies (iv)$ are obvious.

For $(iv) \implies (i)$, consider $p(x) \in \mathbb{Q}[x]$ with $\deg p > 0$ vanishing on $V$. If we call $S'(V)$ the set of indecomposable components of $V^{\otimes m}$ with

$m < \deg p$, we have that $\mathbb{Q}[\mathrm{S}(V)] = \mathbb{Q}[S'(V)]$ thanks to $p(V) = 0$. But the cardinality of $\mathrm{S}(V)$ is $\dim_{\mathbb{Q}} \mathbb{Q}[\mathrm{S}(V)]$ because $\mathbb{Z}[\mathrm{S}(V)] \subseteq K(X)$ is a free abelian group, and the same holds for $S'(V)$ which is finite, hence also $\mathrm{S}(V) = S'(V)$ is finite. □

**Corollary 6.8.** *The following are true:*

*(i)* $E \oplus F$ *is finite if and only if both* $E$ *and* $F$ *are finite.*

*(ii) If* $E$ *and* $F$ *are finite, then* $E \otimes F$ *and* $E^\vee$ *are finite.*

*(iii) A line bundle is finite if and only if it is torsion.*

*Proof.* 1. $\mathrm{S}(E) \cup \mathrm{S}(F) \subseteq \mathrm{S}(E \oplus F)$, hence if $E \oplus F$ is finite $E$ and $F$ are finite, too. On the other hand, if $[E], [F] \in K(X)$ are integral over $\mathbb{Z}$, $[E] + [F] = [E \oplus F]$ is integral, too, and hence $E \oplus F$ is finite.

2. As above, if $[E], [F] \in K(X)$ are integral over $\mathbb{Z}$, $[E] \cdot [F] = [E \otimes F]$ is integral, too, and hence $E \otimes F$ is finite. For $E^\vee$, let $f, g \in \mathbb{N}[x]$ be polynomials such that $f(E) \simeq g(E)$: this implies that

$$f(E^\vee) \simeq f(E)^\vee \simeq g(E)^\vee \simeq g(E^\vee)$$

and hence $E^\vee$ is finite, too.

3. Let $L$ be a line bundle. Clearly, if it is torsion the it is finite. On the other hand, if $L$ is finite, $\mathrm{S}(L)$ is finite, too, and $L^n$ has rank one for every $n$, hence $L^n$ is indecomposable and thus is contained in $\mathrm{S}(L)$. This implies that $L^n \simeq L^m$ for some $n > m$, and hence $L^{n-m} \simeq \mathcal{O}_X$. □

This implies, for example, that finite bundles on $\mathbb{P}^1$ are trivial.

**Definition 6.9.** A vector bundle is *essentially finite* if it is the kernel of a homomorphism of finite bundles. Call $\mathrm{EFin}\, X$ the full subcategory of $\mathrm{Vect}\, X$ whose objects are essentially finite vector bundles.

*Remark* 6.10. Nori here takes a different approach. He *defines* $\pi_1^N(X, x_0)$ using tannakian categories, hence, in order to have an abelian category, he needs to show that finite vector bundles are semistable of degree 0 in order to add subbundles and quotients. With our definitions, a priori, we don't know that $\mathrm{EFin}\, X$ is abelian (i.e. it has kernels and cokernels): we will know it as a corollary of the equivalence with $\mathrm{Rep}_k\, \pi_1^N(X, x_0)$. As a corollary of Nori's theorem, the two definitions of essentially finite bundles coincide.

**Proposition 6.11.** *If $V$ is a representation of $\pi_1^N(X, x_0)$, then $\Phi(V)$ is essentially finite.*

*Proof.* Take $G = \operatorname{Spec} A$ a finite group-scheme such that the action of $\pi_1^N(X, x_0)$ factors through $G$. Thanks to Lemma 2.45, there exist embeddings of $G$-representations $V \hookrightarrow A^n$ and $A^n/V \hookrightarrow A^m$, hence we have an exact sequence of $G$-representations

$$0 \longrightarrow V \longrightarrow A^n \longrightarrow A^m$$

Since $\Phi$ is exact, it is enough to show that $\Phi(A^n)$ and $\Phi(A^m)$ are finite or, equivalently, that $\Phi(A)$ is finite. But to show that $\Phi(A)$ is finite, we can work in $\operatorname{Rep}_k G$: if we find two polynomials $f, g \in \mathbb{N}[x]$ such that $f(A) \simeq g(A)$ as representations, then the same equation will hold for $\Phi(A)$.

So, call $G' = \operatorname{Spec} A'$ where $G' = G$ as a scheme, with $G$ acting on itself by left multiplication and on $G'$ trivially. We have an isomorphism of $G$-schemes $G \times G \to G \times G'$ defined by $(g, h) \mapsto (g, g^{-1}h)$ using the Yoneda Lemma. This defines an isomorphism of representations $A^{\otimes 2} \simeq A \otimes A' \simeq rA$, where $r = \dim A$ and $rA = A^{\oplus r}$. □

## 6.4 Essential surjectivity of $\operatorname{Rep}_k \pi_1 \to \mathrm{EFin}$

The final part of the thesis will be devoted to the proof of the main result.

**Theorem 6.12** (Borne, Nori, Vistoli)**.** *Let $X$ be a pseudo-proper, geometrically connected and geometrically reduced scheme over a field $k$, with a rational point $x_0$. Then there exists an equivalence of neutral tannakian categories between $\operatorname{Rep}_k \pi_1^N(X, x_0)$ and $\mathrm{EFin}\, X$ sending the forgetful functor to the fibre functor over $x_0$.*

We have already proved that $\Phi$ is a fully faithful functor $\operatorname{Rep}_k \pi_1^N(X, x_0) \to \mathrm{EFin}\, X$, we are left with proving that it is also essentially surjective. Since $\Phi$ is exact, it is enough to show that finite bundles belong to the essential image. We begin by proving that, for every vector bundle $E$ of rank $n$, there exists a $\mathrm{GL}_n$-torsor $P$ such that the pullback of $E$ to $P$ is the $\mathrm{GL}_n$-equivariant sheaf given by the standard representation of $\mathrm{GL}_n$ on $k^n$. Then, we will show that we can reduce ourselves to a finite subgroup of $\mathrm{GL}_n$ if $E$ is finite.

Let $V = k^n$ be the standard representation of $\mathrm{GL}_n$.

**Lemma 6.13.** *Consider the functor* $\mathrm{Fr}_E : \mathrm{Sch}\,/k^{\mathrm{op}} \to \mathrm{Set}$*, called the functor of frames of $E$, sending a scheme $S$ to the set of cartesian diagrams of the form*

$$\begin{tikzcd} V \otimes \mathcal{O}_S \arrow[r] \arrow[d, maps to] & E \arrow[d, maps to] \\ S \arrow[r] & X \end{tikzcd}$$

*defining a morphism of vector bundles. There exists a $\mathrm{GL}_n$-torsor $j : P \to X$, called the bundle of frames, representing $\mathrm{Fr}_E$. Moreover, $j^*E \simeq V \otimes \mathcal{O}_P$ as equivariant sheaves, where $\mathrm{GL}_n$ acts on $j^*E$ with the standard action on pullbacks along invariant maps and on $V \otimes \mathcal{O}_P$ with the standard representation on $V = k^n$.*

*Proof.* Consider the sheaf of $\mathcal{O}_X$-algebras $\mathcal{A} = \mathrm{Sym}\left(E^\vee \otimes V\right)$, where $E^\vee \otimes V$ can be thought as $\underline{\mathrm{Hom}}(E, V \otimes \mathcal{O}_X)$.

Since $E$ is locally free of rank $n$, we can define locally a section $\det : \mathcal{O}_X \to \mathcal{A}$ and consider the localization $\mathcal{A}_{\det}$. The local section det depends on the local trivialization of $E$, but only up to an invertible section of $\mathcal{A}$, hence $\mathcal{A}_{\det}$ is well defined globally. Now, call $P$ the relative spectrum $\mathrm{Spec}\,\mathcal{A}_{\det}$ and $j : P \to X$ the canonical projection.

I claim that $P$ represent $\mathrm{Fr}_E$. In fact, take a morphism $s : S \to P$ and call $f$ the composition $j \circ s : S \to X$. Since $P = \mathrm{Spec}\,\mathcal{A}_{\det}$, $s$ corresponds to a morphism of $\mathcal{O}_S$-algebras $f^*\mathcal{A}_{\det} \to \mathcal{O}_S$. Since $\mathcal{A} = \mathrm{Sym}\left(E^\vee \otimes V\right)$, the composed map of $\mathcal{O}_S$-algebras $f^*\mathcal{A} \to \mathcal{O}_S$ corresponds to an $\mathcal{O}_S$-linear map $f^*E^\vee \otimes V \to \mathcal{O}_S$, which in turn corresponds to an $\mathcal{O}_S$-linear map $\lambda : V \otimes \mathcal{O}_S \to f^*E$. We have thus a commutative diagram

$$\begin{tikzcd} V \otimes \mathcal{O}_S \arrow[r] \arrow[d, maps to] & E \arrow[d, maps to] \\ S \arrow[r] & X \end{tikzcd}$$

that is cartesian if and only if $\lambda$ is an isomorphism.

This is a local problem, hence we may suppose $E = \mathcal{O}_X^n$. We have that $\lambda$ is defined by a morphism

$$f^*E^\vee \otimes V = \mathcal{O}_S^{n\vee} \otimes V \to \mathcal{O}_S$$

that factors through $f^*\mathcal{A}_{\det} = \underline{\mathrm{Hom}}(\mathcal{O}_S^n, V \otimes \mathcal{O}_S)_{\det}$ if and only if $\lambda$ is invertible, as desired.

Now, $g \in \mathrm{GL}_n(S)$ acts on $\mathrm{Fr}_E$ by sending a cartesian diagram

$$\begin{tikzcd} V \otimes \mathcal{O}_S \arrow[r] \arrow[d, mapsto] & E \arrow[d, mapsto] \\ S \arrow[r] & X \end{tikzcd}$$

to the composition

$$\begin{tikzcd} V \otimes \mathcal{O}_S \arrow[r, "g^{-1}"] \arrow[dr, mapsto] & V \otimes \mathcal{O}_S \arrow[r] \arrow[d, mapsto] & E \arrow[d, mapsto] \\ & S \arrow[r] & X \end{tikzcd}$$

By the Yoneda Lemma, this induces an action of $\mathrm{GL}_n$ on $P$ such that $P \to X$ is $\mathrm{GL}_n$-invariant. Being locally free of fixed rank, $\operatorname{Spec} \mathcal{A} \to X$ is obviously faithfully flat and affine, hence its localization $P \to X$ is still flat and affine. However, localizing we may be unlucky and take away entire fibers of $\operatorname{Spec} \mathcal{A} \to X$, losing surjectivity. To check that this is not the case, take a point $q : \operatorname{Spec} \Omega \to X$ for some field $\Omega$ and consider the cartesian diagram

$$\begin{tikzcd} q^*E \arrow[r] \arrow[d] & E \arrow[d] \\ \operatorname{Spec} \Omega \arrow[r] & X \end{tikzcd}$$

Since $E$ is locally free, $q^*E \simeq \Omega^n$, and since $P$ represents $\mathrm{Fr}_E$, this defines a point $\operatorname{Spec} \Omega \to P$ over $q$.

To prove that $P \to X$ is a torsor, we only need to show that $\mathrm{GL}_n \times P \to P \times_X P$ is an isomorphism: this becomes trivial if we use the Yoneda Lemma and show that $\mathrm{GL}_n \times \mathrm{Fr}_E \simeq \mathrm{Fr}_E \times_X \mathrm{Fr}_E$.

Hence we are left with proving that $j^*E$ is isomorphic to $V \otimes \mathcal{O}_P$ and that the induced structure of $\mathrm{GL}_n$-sheaf on $j^*E$ corresponds to the one induced by the standard representation of $\mathrm{GL}_n$ on $\mathcal{O}_P^n$.

Since $P$ represents $\mathrm{Fr}_E$, consider the cartesian diagram associated to $\mathrm{id}_P$:

$$\begin{tikzcd} V \otimes \mathcal{O}_P \arrow[r] \arrow[d, mapsto] & E \arrow[d, mapsto] \\ P \arrow[r] & X \end{tikzcd}$$

The lower horizontal arrow is $j \circ \mathrm{id}_P = j$, hence $V \otimes \mathcal{O}_P \simeq j^*E$. Finally, we just have to unwind the definitions to check that the action on $V \otimes \mathcal{O}_P$ as pullback of $E$ is exactly the action defined by the standard representation of $\mathrm{GL}_n$. Hence, let $s : S \to P$ be a morphism, $\eta \mapsto S$ a sheaf, $\alpha : \eta \to j^*E$ a morphism of sheaves over $s$ and $g$ an element of $\mathrm{GL}_n(S)$. We have a commutative diagram

$$\begin{tikzcd}
\eta \arrow[r, "\alpha"] \arrow[dr, mapsto] & s^*V \otimes \mathcal{O}_P \arrow[r] \arrow[d, mapsto] & V \otimes \mathcal{O}_P \arrow[d, mapsto] \\
 & S \arrow[r, "s"] & P
\end{tikzcd}$$

The action of $g$ on $\alpha$ induced by the pullback of $E$ is the composition

$$\begin{tikzcd}
\eta \arrow[r, "\alpha"] \arrow[dr, mapsto] & s^*V \otimes \mathcal{O}_P = (j \circ s)^*E = (j \circ gs)^*E \arrow[r] \arrow[d, mapsto] & V \otimes \mathcal{O}_P = j^*E \arrow[d, mapsto] \\
 & S \arrow[r, "gs"] & P
\end{tikzcd}$$

while the action induced by the standard representation of $\mathrm{GL}_n$ is

$$\begin{tikzcd}
\eta \arrow[r, "\alpha"] \arrow[dr, mapsto] & s^*V \otimes \mathcal{O}_P \simeq V \otimes \mathcal{O}_S \arrow[r, "g"] \arrow[d, mapsto] & V \otimes \mathcal{O}_S \simeq (gs)^*V \otimes \mathcal{O}_P \arrow[r] \arrow[dl, mapsto] & V \otimes \mathcal{O}_P \arrow[d, mapsto] \\
 & S \arrow[rr, "gs"] & & P
\end{tikzcd}$$

and, by definition, $gs$ is the unique morphism $S \to X$ such that

$$V \otimes \mathcal{O}_S \xrightarrow{\sim} (gs)^*V \otimes \mathcal{O}_P \xrightarrow{=} (j \circ gs)^*E \xrightarrow{=} (j \circ s)^*E$$

is equal to

$$V \otimes \mathcal{O}_S \xrightarrow{g^{-1}} V \otimes \mathcal{O}_S \xrightarrow{\sim} s^*V \otimes \mathcal{O}_P \xrightarrow{=} (j \circ s)^*E.$$

□

So, we have shown that there is a $\mathrm{GL}_n$-torsor such that $E$ is in the essential image of the functor $\mathrm{Rep}_k \mathrm{GL}_n \to \mathrm{Vect}\, X$ defined by $P$. The problem is that $\mathrm{GL}_n$ is not finite: but if we find a finite subgroup $G \subseteq \mathrm{GL}_n$ and a reduction of structure group of $P$ to $G$, then $E$ is in the essential image of the induced functor $\mathrm{Rep}_k G \to \mathrm{Vect}\, X$. Thanks toProposition 4.11, it is enough to find a finite subgroup $G \subseteq \mathrm{GL}_n$ and a $\mathrm{GL}_n$-equivariant morphism $P \to \mathrm{GL}_n / G$ when $E$ is finite.

Let $f, g \in \mathbb{N}[t]$ be polynomials such that $f(E) \simeq g(E)$. Since the Krull-Schmidt-Remak theorem holds, we may suppose that $\deg f \neq \deg g$. Now, set $V = k^n$ and call $I$ the variety of isomorphisms $f(V) \simeq g(V)$: this is simply the affine variety $\mathrm{Hom}(f(V), g(V))_{\det}$, where $\det \in \mathrm{H}^0(\mathrm{Hom}(f(V), g(V)))$ is defined up to an invertible constant, irrelevant in the construction of the localization. If we fix a basis for $V$, and hence of $f(V)$ and $g(V)$, we get an isomorphism $I \simeq \mathrm{GL}_N$ where $N = f(n) = g(n)$. The scheme $I$ represents the functor sending a scheme $S$ to the set of commutative diagrams of the form

$$\begin{tikzcd} f(V) \otimes \mathcal{O}_S \arrow[rr, "\sim"] \arrow[dr] & & g(V) \otimes \mathcal{O}_S \arrow[dl] \\ & S & \end{tikzcd}$$

where the upper arrow is an isomorphism, and the proof is analogous to the one of Lemma 6.13. Moreover, on $I$ there is a natural action of $\mathrm{GL}_n$: a point $\sigma \in \mathrm{GL}_n(S)$ acts naturally both on $f(V) \otimes \mathcal{O}_S$ and $g(V) \otimes \mathcal{O}_S$ with the action on $V$, hence on $\lambda : f(V) \otimes \mathcal{O}_S \to g(V) \otimes \mathcal{O}_S$ as

$$g \cdot \lambda = g \circ f \circ g^{-1}.$$

**Lemma 6.14.** *The isomorphism $f(E) \simeq g(E)$ induces a $\mathrm{GL}_n$-equivariant morphism $\psi : P \to I$.*

*Proof.* The pullback of $f(E) \simeq g(E)$ to $P$ is an isomorphism

$$\lambda : f(V) \otimes \mathcal{O}_P \to g(V) \otimes \mathcal{O}_P$$

that is $\mathrm{GL}_n$-equivariant with respect to the action on both $V$ and $\mathcal{O}_P$. This in turn yields to a morphism $\varphi : P \to I$ that is $\mathrm{GL}_n$-equivariant. If $s \in P(S)$ and $\sigma \in \mathrm{GL}_n(S)$, we want to compare $\varphi(\sigma s)$ and $\sigma\varphi(s)$ as isomorphisms $f(V) \otimes \mathcal{O}_S \simeq g(V) \otimes \mathcal{O}_S$. Using the Yoneda Lemma, if $T$ is a scheme, $t$ a point of $S(T)$ and $\alpha : \eta \to f(V) \otimes \mathcal{O}_T$ a morphism of sheaves over $T$, we have

$$\varphi(\sigma s)(\alpha, t) = \lambda(\alpha, (\sigma s)t)$$

and

$$\sigma\varphi(s)(\alpha,t) = \sigma(\varphi(s)(\sigma^{-1}\alpha,t)) = \sigma(\lambda(\sigma^{-1}\alpha,st)) = \lambda(\alpha,(\sigma s)t)$$

where the last equality comes from the fact that $\lambda$ is $\mathrm{GL}_n$-equivariant with respect to the action on both components. $\square$

**Proposition 6.15.** *The quotient $\varphi : \mathrm{GL}_n \backslash I$ exists as an affine categorical quotient, and $I \to \mathrm{GL}_n \backslash I$ is submersive, surjective and separates closed, $\mathrm{GL}_n$-invariant subsets. Moreover, the quotient is stable under faithfully flat base change.*

*Proof.* Thanks to [MFK94, Ch.2, §.2, Theorem 1.1], $\mathrm{GL}_n \backslash I = \operatorname{Spec} \mathrm{H}^0(I)^{\mathrm{GL}_n}$ exists as an affine categorical quotient, $\varphi : I \to \mathrm{GL}_n \backslash I$ is dominant, sends invariants closed subsets to closed subsets and separates closed, invariant subsets. In particular, $\varphi(I)$ is closed and dense, hence $\varphi$ is surjective and $\mathrm{GL}_n \backslash I$ has the quotient topology. $\square$

In Lemma 3.39, we proved that $X$ is a geometric quotient of $P$ for the action of $\mathrm{GL}_n$, and so the composition $P \to I \to \mathrm{GL}_n \backslash I$ passes to the quotient $X = \mathrm{GL}_n \backslash P \to \mathrm{GL}_n \backslash I$. Moreover, since $X$ is pseudo-proper and $\mathrm{GL}_n \backslash I$ is affine, the map $X \to \mathrm{GL}_n \backslash I$ splits as $X \to q \to \mathrm{GL}_n \backslash I$ where $q$ is a rational point. We have thus the following $\mathrm{GL}_n$-equivariant diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \varphi^{-1}(q) \arrow[r] \arrow[d] & I \arrow[d] \\ X \arrow[r] & q \arrow[r] & \mathrm{GL}_n \backslash I \end{tikzcd}$$

Now, the action of $\mathrm{GL}_n$ on $\varphi^{-1}(q)$ induces an action on $\varphi^{-1}(q)_{\mathrm{red}}$. In fact, we have a morphism $\mathrm{GL}_n \times \varphi^{-1}(q)_{\mathrm{red}} \to \varphi^{-1}(q)$. To show that it descends to a morphism $\mathrm{GL}_n \times \varphi^{-1}(q)_{\mathrm{red}} \to \varphi^{-1}(q)_{\mathrm{red}}$, it is enough to show that $\mathrm{GL}_n \times \varphi^{-1}(q)_{\mathrm{red}}$ is reduced: this is true because $\mathrm{GL}_n$ is an open subset of $\mathbb{A}^{n^2}$ and, if $U \subseteq \varphi^{-1}(q)_{\mathrm{red}}$ is an affine open subset,

$$\mathbb{A}^{n^2} \times U = \operatorname{Spec}\left(k[x_1,\dots,x_{n^2}] \otimes \mathrm{H}^0(U)\right)$$

is reduced.

We also have that $\psi : P \to I$ factors through $\varphi^{-1}(q)_{\mathrm{red}}$ because $P$ is reduced. In fact, following the construction of Lemma 6.13, $P$ is an open subscheme of the relative spectrum $\operatorname{Spec} \mathcal{A}$, where $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebras such that, for every $p \in X$, $\mathcal{A}_p \simeq \mathcal{O}_{X,p}[x_1,\dots,x_s]$ for some $s$. Since $X$ is reduced, this implies that $\mathcal{A}$ is reduced, too.

Hence, we get the following $\mathrm{GL}_n$-equivariant diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \varphi^{-1}(q)_{\mathrm{red}} \arrow[r] \arrow[d] & I \arrow[d] \\ X \arrow[r] & q \arrow[r] & \mathrm{GL}_n \backslash I \end{tikzcd}$$

If we show that $\varphi^{-1}(q)_{\mathrm{red}} \simeq \mathrm{GL}_n /G$, we have finished. Take a rational point $p \in P(k)$ over $x_0$ ($p$ exists because $P$ represents $\mathrm{Fr}_E$ and $E$ is locally free) and call $G \subseteq \mathrm{GL}_n$ the stabilizer of $\varphi(p) \in I(k)$: we claim that $G$ is finite.

**Lemma 6.16.** *Rational points of $I$ have finite stabilizers in* $\mathrm{GL}_n$*.*

*Proof.* Since taking the stabilizer commutes with base change to $\bar{k}$, we may suppose $k = \bar{k}$.

For every $H$ subgroup of $G$, $f(V)$ is isomorphic to $g(V)$ as a representation of $H$. If $G$ has positive dimension, it contains either a copy of $\mathbb{G}_a$ or of $\mathbb{G}_m$ thanks to [Spr98, Lemma 6.3.4] (here we are using $k = \bar{k}$). Hence it is enough to show that for $H = \mathbb{G}_a$ or $H = \mathbb{G}_m$ and $V$ a non trivial representation of $H$, $f(V) \not\simeq g(V)$.

For any representation $W$, define the $\delta$-invariant $\delta(W)$ as follows. Since $H = \mathbb{G}_m$ or $H = \mathbb{G}_a$, $\mathrm{H}^0(H) \subseteq k[t^{\pm 1}]$. Fix a basis of $W$ such that $\mathrm{GL}(W) \simeq \mathrm{GL}_{\dim W}$ and write the action of $H$ as an invertible matrix $h(t)$ whose entries polynomials in $k[t^{\pm 1}]$. It is well defined the degree of a polynomial in $k[t^{\pm 1}]$ as the maximum of the exponents of its monomials. Call $\delta(W)$ the maximum degree of the entries in $h(t)$. Clearly, $\delta(W)$ does not depend on the basis: if we change basis it cannot increase because we are only doing linear combinations, but then it can't decrease too, because we can go back to the first basis with another change. Furthermore, $\delta(W \oplus W') = \max\{\delta(W), \delta(W')\}$ and $\delta(W \otimes W') = \delta(W) + \delta(W')$.

Hence, $\delta(f(V)) = \deg f \cdot \delta(V) \neq \deg g \cdot \delta(V) = \delta(g(V))$, since $\deg f \neq \deg g$ and $\delta(V) \neq 0$ because $V$ is a non trivial representation. □

Since the stabilizer of $p$ is finite, Theorem 3.45 gives us the quotient $\mathrm{GL}_n /G$, and $p$ defines a $\mathrm{GL}_n$-equivariant map $\mathrm{GL}_n /G \to \varphi^{-1}(q)_{\mathrm{red}}$. We have already shown the existence of a $\mathrm{GL}_n$-equivariant map $P \to \varphi^{-1}(q)_{\mathrm{red}}$, if we prove that $\mathrm{GL}_n /G \to \varphi^{-1}(q)_{\mathrm{red}}$ is an isomorphism we have finished.

**Lemma 6.17.** *In order to show that* $\mathrm{GL}_n /G \to \varphi^{-1}(q)_{\mathrm{red}}$ *is an isomorphism, we may suppose $k = \bar{k}$.*

*Proof.* Let us suppose that $\mathrm{GL}_{n,\bar{k}}/G_{\bar{k}} \to \varphi_{\bar{k}}^{-1}(q_{\bar{k}})_{\mathrm{red}}$ is an isomorphism. Since $\operatorname{Spec}\bar{k} \to \operatorname{Spec}k$ is faithfully flat, $\mathrm{GL}_n/G \to \varphi^{-1}(q)_{\mathrm{red}}$ is an isomorphism if and only if $(\mathrm{GL}_n/G)_{\bar{k}} \to \varphi^{-1}(q)_{\bar{k}}$ is an isomorphism. Hence, it is enough to show that $(\mathrm{GL}_n/G)_{\bar{k}} = \mathrm{GL}_{n,\bar{k}}/G_{\bar{k}}$ and $\varphi^{-1}(q)_{\bar{k}} = \varphi_{\bar{k}}^{-1}(q_{\bar{k}})$.

We have $\mathrm{GL}_n/G = \operatorname{Spec}\mathrm{H}^0(\mathrm{GL}_n)^G$, hence we need to show that the natural map $\mathrm{H}^0(\mathrm{GL}_{n,\bar{k}})^{G_{\bar{k}}} \to \mathrm{H}^0(\mathrm{GL}_n)\otimes\bar{k}$ is an isomorphism. We have an equalizer of $k$-modules

$$0 \to \mathrm{H}^0(\mathrm{GL}_n)^G \to \mathrm{H}^0(\mathrm{GL}_n) \rightrightarrows \mathrm{H}^0(\mathrm{GL}_n)\otimes\mathrm{H}^0(G)$$

and, since $\operatorname{Spec}\bar{k} \to \operatorname{Spec}k$ is faithfully flat,

$$0 \to \mathrm{H}^0(\mathrm{GL}_n)^G\otimes\bar{k} \to \mathrm{H}^0(\mathrm{GL}_{n,\bar{k}}) \rightrightarrows \mathrm{H}^0(\mathrm{GL}_{n,\bar{k}})\otimes_{\bar{k}}\mathrm{H}^0(G_{\bar{k}})$$

is an equalizer, too, and hence $\mathrm{H}^0(\mathrm{GL}_{n,\bar{k}})^{G_{\bar{k}}} = \mathrm{H}^0(\mathrm{GL}_n)\otimes\bar{k}$.

For $\varphi^{-1}(q)_{\bar{k}} = \varphi_{\bar{k}}^{-1}(q_{\bar{k}})$, consider the following diagram:

$$\begin{tikzcd}
& \varphi_{\bar{k}}^{-1}(q_{\bar{k}}) \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & I_{\bar{k}} \arrow[dl] \arrow[dd] \\
\varphi^{-1}(q) \arrow[rr] \arrow[dd] & & I \arrow[dd] & \\
& q_{\bar{k}} \arrow[rr, dotted] \arrow[dl, dotted] & & \mathrm{GL}_{n,\bar{k}}\backslash I_{\bar{k}} \arrow[dl] \\
q \arrow[rr] & & \mathrm{GL}_n\backslash I &
\end{tikzcd}$$

The square on the right is cartesian thanks to Proposition 6.15 and the fact that $\operatorname{Spec}\bar{k} \to \operatorname{Spec}k$ is faithfully flat. This implies that also the square on the left is cartesian, and hence $\varphi^{-1}(q)_{\bar{k}} = \varphi_{\bar{k}}^{-1}(q_{\bar{k}})$. □

From now on, suppose $k = \bar{k}$. Thanks to Proposition 3.42.(iv), it is enough to show that $\varphi^{-1}(q)$ is, set-theoretically, the orbit set of $p$.

**Lemma 6.18.** *Orbit sets of rational points of $I$ are closed.*

*Proof.* Let $s \in I(k)$ be a rational point. Thanks to Proposition 3.42.(i), $\mathrm{GL}_n s$ is open in $\overline{\mathrm{GL}_n s}$: let us suppose that they are different.

Take a closed point $s' \in \overline{\mathrm{GL}_n s} \setminus \mathrm{GL}_n s$, $s'$ is rational because $k = \bar{k}$. Since $\overline{\mathrm{GL}_n s} \setminus \mathrm{GL}_n s$ is $\mathrm{GL}_n$-invariant,

$$\mathrm{GL}_n s' \subseteq \overline{\mathrm{GL}_n s} \setminus \mathrm{GL}_n s,$$

but this is absurd because both $\mathrm{GL}_n s$ and $\mathrm{GL}_n s'$ have dimension $n^2$ thanks to Proposition 3.42.(iii) and Lemma 6.16. □

Now, $\varphi^{-1}(q)$ is closed and contains $\mathrm{GL}_n p$ which is closed, too. We also know that $\varphi^{-1}(q)$ is Jacobson thanks to [Bou64, V.3.4, Theorem 3], hence $\varphi^{-1}(q) \setminus \mathrm{GL}_n p$ contains a closed point $p'$ if it is nonempty. But then $p'$ is rational and $\mathrm{GL}_n p'$ is closed: this is absurd, because $\varphi$ separates closed, invariant subsets.

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Nel ventre tuo si raccese l’amore,
per lo cui caldo ne l’etterna pace
così è germinato questo fiore.
DANTE, *Par.* XXXIII, 7-9